# Smooth Rational Surfaces of Degree Eleven and Sectional Genus Eight in the Projective Fivespace.

by

Abdul Moeed Mohammad

Thesis for the degree of

**Master in Mathematics**

(Master of Science)

Department of Mathematics
Faculty of Mathematics and Natural Sciences
University of Oslo

May 2010

# Contents

# 1 Introduction.

A natural task in any scientific discipline is to organize objects similar to each other and thus classifying the objects relative to a similarity. In mathematics the similarity is often found in numerical invariants of the objects and the field of algebraic geometry does not differ from this viewpoint. Indeed, to any projective scheme $X$ we can associate a numerical polynomial, the Hilbert polynomial, which encodes both instrinsic invariants and invariants relative to some projective embedding of $X$. In thesis we ask ourselves the following question in a very concrete setting.

**Question.** Given a numerical polynomial

$$P(n) = \sum_i a_i \binom{n}{i},$$

does there exist a projective scheme $X$ with $P(n)$ as its Hilbert polynomial?

In this thesis we restrict ourselves to linearly normal smooth rational surfaces in $\mathbb{P}^5$. Let $S$ be any such surface. Arrondo, Sols [AS89] and Gross [Gro93] have classified surfaces contained in smooth quadrics. We have chosen the extrinsic invariants, the degree of $S$ and sectional genus of $S$, such that $S$ may be contained in a quadric, not necessarily smooth. Our choice has lead us to consider rational surfaces with speciality 1. These are surfaces obtained by blowing up the projective plane at finitely many points in a special position. In terms of Question, we will be determining the existence of rational surfaces $\subset \mathbb{P}^5$ with

$$P(n) = 11\binom{n}{2} + 4n + 1$$

as their Hilbert polynomial. Our approach is constructive and our techniques are inspired by classification of surfaces in $\mathbb{P}^4$ done in the late 1980s, such as in [Ale88] or [Ran88]. We answer the Question in the following order.

**Step 1.** Find finitely many linear systems with the Hilbert polynomial above.
**Step 2.** Show the non-existence of possiblities obtained in Step 1.
**Step 3.** Use the possibilities in Step 1 and explicitly construct a smooth rational surface with the Hilbert polynomial above.

The thesis is organized as the Steps above.
In Chapter 3 we do Step 1 by using results on the adjunction mapping, due to [Som79], [Som80], [SV87] and [VdV79]. We end Chapter 3 with doing Step 2.
In Chapter 4 we discuss a general strategy for Step 3 and how the strategy breaks with the construction of non-special surfaces done in articles such as [CF93] and [CH97]. We end Chapter 4 by doing weak versions of Step 2 for some explicit embeddings.
In Chapter 5 we do Step 3 for a particular case.

**Results.**

Let $S$ be a linearly normal smooth rational surface of degree 11 and sectional genus 8 embedded in $\mathbb{P}^5$ by a very ample complete linear system $|H|$. In terms of linear systems we show that either the complete linear system $|H + nK_S|$ maps $S$ to a curve for some $n > 0$, or $|H + nK_S|$ induces a $2:1$ map for some $n > 1$, or $(S, H)$ is one of the following:

| $S$ | $H$ |
|---|---|
| $\tilde{\mathbb{P}}^2(x_1, .., x_{19})$ | $6L - \sum_{i=1}^{2} 2E_i - \sum_{j=3}^{19} E_j.$ |
| $\tilde{\mathbb{P}}^2(x_1, .., x_{17})$ | $7L - \sum_{i=1}^{7} 2E_i - \sum_{j=3}^{17} E_j.$ |
| $\tilde{\mathbb{P}}^2(x_1, .., x_{16})$ | $9L - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{8} 2E_j - \sum_{k=9}^{16} E_k.$ |
| $\tilde{\mathbb{P}}^2(x_1, .., x_{15})$ | $10L - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{15} E_j.$ |

Our Main Theorem is a positive answer to the Question in the previous page. We achieve this by explicitly constructing a smooth rational surface $S$ in the following manner.

Given 5 points $x_1, .., x_5 \in \mathbb{P}^2$ in general position it is possible to choose 12 points $y_1, y_2, z_1, .., z_{10} \in \mathbb{P}^2$ such that if

$$\pi : S \longrightarrow \mathbb{P}^2$$

is the morphism obtained by blowing up $x_1, .., x_5, y_1, y_2, z_1, .., z_{10}$, where $E_i := \pi^{-1}(x_i)$, $F_i := \pi^{-1}(y_i)$, $G_i := \pi^{-1}(z_i)$ and $l \subset \mathbb{P}^2$ is a line then

$$|H| = |7\pi^* l - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} 2F_j - \sum_{k=1}^{10} G_k|$$

is very ample on $S$ and has projective dimension $\dim |H| = 5$. The points $y_1, y_2, z_1, .., z_{10}$ can be chosen such that there are two curves

$$6l - \sum 2x_i - \sum y_i - \sum z_i$$

$$4l - \sum x_i - \sum y_i - \sum z_i$$

in $\mathbb{P}^2$ sharing common tangent directions at $y_1, y_2$ and meeting transversally at $z_1, .., z_{10}$.

**Acknowledgements.**

First and foremost, I would like to thank my advisor Professor Kristian Ranestad for interesting discussions and his continuous encouragement. He has impressed me countless times with his intuition, his ability to teach and his dedication.
Many of the ideas in this thesis originate from a seminar held by Ranestad about classification of smooth projective surfaces in Fall 2009.

I would also like to thank the rest of the Algebraic Geometry group at the University of Oslo. In particular, I would like to thank Andrea Hofmann, Robin Bjørnetun Jacobsen and John Christian Ottem for always being interested in discussing both mathematics and teaching.

At a personal level I would like to thank my parents, Abdur and Shamim.

## 2 Basic results.

We use standard definitions as in [Har77]. Our notation is as in [Bea96].

A *surface* will mean a smooth projective scheme over an algebraically closed field of characteristic 0 and will always be denoted by $S$. By the *sectional genus* $\pi_S$ we mean the genus of a general hyperplane section of $S$. We write $\equiv$ for linear equivalence and $\simeq$ for isomorphism. The symbol $\varphi_A$ will always refer to the map associated to a complete linear system $|A|$ on $S$, given that $\varphi_A$ exists.

A *rational* surface $S$ is a surface equipped with a birational morphism

$$\pi : S \longrightarrow \mathbb{P}^2.$$

Due to a famous theorem, Theorem V.5.5 in [Har77], the birational morphism $\pi$ is a finite composition of monoidal transformations centered at $x_1, .., x_r \in \mathbb{P}^2$. We will therefore write $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_r)$. Furthermore, $\operatorname{Pic}\tilde{\mathbb{P}}^2(x_1, .., x_r) \simeq \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus ... \oplus \mathbb{Z}E_r$, where $E_i = \pi^{-1}(x_i)$ is an exceptional divisor on $S$ and $L = \pi^* l$ for some line $l \subset \mathbb{P}^2$.
By the *type* of a divisor class $D \equiv aL - \sum^r b_i E_i$ we will mean the shorthand notation $[a; \max\{b_i\}^{u_0}, ..., \min\{b_i\}^{u_v}]$, where $\sum_0^v u_i = r$ and $u_j = \#\{k \mid b_k = \max\{b_i\} - j\}$.
We denote the Hirzebruch surfaces by $\mathbb{F}_e$, where $e \geq 0$. We write $B$ for the class of a section with $B^2 = e$ and write $F$ as a fiber in the ruling.

A *curve* on $S$ will always mean an effective divisor on $S$, not neccessarily smooth. For smooth curves on $S$ the following theorem will be useful.

**Theorem (Adjuction formula).** *Let $C$ be a smooth curve on a surface $S$. Then:*
*(1).* $\omega_C \simeq \mathscr{O}_C(C + K_S)$.
*(2).* $2p_a(C) - 2 = C.(C + K_S)$.
*(3).* *If $S$ is rational and $C \equiv aL - \sum b_i E_i$, then* $p_a(C) = \binom{a-1}{2} - \sum \binom{b_i}{2}$.

*Proof.* See Theorem 1.6.3 in [BPVdV84] for a proof of (1). The statement (2) follows from taking the degree of (1). The statement in (3) follows by rearranging (2) and recalling that $K_S \equiv -3L + \sum E_i$ since $S$ is rational. □

An immediate corollary of the adjunction formula is the *addition formula for the arithmetic genus*, for curves $C$ and $D$, given by

$$p_a(C + D) = p_a(C) + p_a(D) + C.D - 1.$$

Let $X$ be either a curve or a surface. We will say that a divisor $D$ is *special on $X$* if $h^1(\mathscr{O}_X(D)) > 0$. Otherwise we say that $D$ is *non-special on $X$*. We will often be using *Serre duality* in the following form as in Corollary III.7.7. in [Har77], namely

$$H^i(\mathscr{O}_X(D)) \simeq H^{\dim X - i}(\mathscr{O}_X(K_X - D)).$$

The following will be used to compute the dimensions of cohomology groups.

**Theorem (Riemann-Roch).** *Let $D$ be a divisor on $X$. Then:*

*(1).* *If* $\dim X = 1$, *then* $\chi(\mathscr{O}_X(D)) = D^2 + 1 - p_a(D)$.

*(2).* *If* $\dim X = 2$, *then* $\chi(\mathscr{O}_X(D)) = \frac{1}{2}D.(D - K_S) + \chi(\mathscr{O}_S)$.

*(3).* *If $X$ is a rational surface and* $D \equiv aL - \sum b_i E_i$, *then* $\chi(\mathscr{O}_S(D)) = \binom{a+2}{2} - \sum \binom{b_i+1}{2}$.

*Proof.* See Theorem IV.1.3 in [Har77] for a proof of (1). See Theorem V.1.6 in [Har77] for a proof of (2). The statement in (3) follows directly by combining (2) with $\chi(\mathscr{O}_S) = 1$ and with $K_S \equiv -3L + \sum E_i$. □

We will use the following to bound $K_S^2$.

**Theorem (Hodge index inequality).** *Suppose $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_r)$ and suppose $H$ is ample on $S$, then*

$$(9 - r)H^2 \leq (H.K_S)^2.$$

*Proof.* Note that $H.((H^2)K_S - (H.K_S)H) = 0$. Then the Hodge index theorem, Theorem V.1.9 in [Har77], yields that $((H^2)K_S - (H.K_S)H)^2 \leq 0$. Since $H^2 > 0$, the latter gives us $(K_S^2)(H^2) \leq (H.K_S)^2$. Then we may use that $K_S^2 = 9 - r$. □

For the vanishing of cohomology groups we need the next theorem.

**Theorem (Kodaira vanishing theorem).** *Suppose $S$ is a smooth surface and $H$ is very ample on $S$. Then $H^1(\mathscr{O}_S(H + K_S)) = 0$.*

*Proof.* See Theorem IV.8.6 in [BPVdV84]. □

A surface $S$ is said to *degenerated* if $S$ is contained within a hyperplane, and *non-degenerated* otherwise. The following inequality holds for non-degenerated surfaces.

**Proposition 1.** *Suppose $S \subset \mathbb{P}^n$ is a non-degenerated surface. Then* $\deg S \geq n - 1$.

*Proof.* See Proposition 0 in [EH87]. □

A classical result due to Del Pezzo classifies surfaces whenever equality occurs in Proposition 1.

**Theorem 2.** *Suppose $S \subset \mathbb{P}^n$ is a surface and* $\deg S = n-1$. *Then either $S$ is a minimal rational scroll $\subset \mathbb{P}^n$ or $S$ is the Veronese surface.*

*Proof.* See Theorem 1 in [EH87]. □

Let $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_r)$. By a *general position* (resp. *configuration*) of the points $x_1, .., x_r$ we shall mean that there exist no plane curve of degree $d$ with $m_i := \mathrm{mult}_{x_i}$ such that $\binom{d+2}{2} \leq \sum_{i=1}^{r} \binom{m_i+1}{2}$, for all $d \in \mathbb{N}$ except $(d, r) \in \{(2, 2), (4, 5)\}$. Otherwise, we will say that the points $x_1, .., x_r$ are in *special position* (resp. *configuration*).

By an *open condition* on the choice of $r$ distinct points $x_1, .., x_r \in \mathbb{P}^2$ we shall mean that there exists an open set in the *$r$-th configuration space* satisfying the condition. Otherwise, we will mean a *closed condition* on the choice of points $x_1, .., x_r \in \mathbb{P}^2$.

# 3 Linear systems via adjunction.

In this chapter we will state some results on the adjunction mapping due to Sommese and Van de Ven. These results will be used to construct an algorithm for computing possibilities a projective embedding of rational surfaces can take. In particular we prove some bounds and relations between various invariants relative to adjunction mappings of rational surfaces. We will then employ our algorithm using our bounds and relations to describe all possibilities the complete linear system of a very ample divisor embedding a rational surface of degree 11 and sectional genus 8 into $\mathbb{P}^5$ can take.

## 3.1 Adjunction theory.

Let $i : S \hookrightarrow \mathbb{P}^n$ be an embedding with $\mathscr{L} = i^*\mathscr{O}_{\mathbb{P}^n}(1)$ as the associated very ample line bundle on $S$ and let $H$ be a divisor on $S$ such that $\mathscr{L} = \mathscr{O}_S(H)$. Classical adjunction theory, dating back to the Italian School of Geometry, is mainly interested in studying the pair $(S, \mathscr{L})$ through studying the pair $(S, \mathscr{L} \otimes \omega_S)$ instead. The line bundle $\mathscr{L} \otimes \omega_S$ is called the *adjunction bundle of* $\mathscr{L}$ *on* $S$ and it's associated projective mapping $\varphi_{\mathscr{L}\otimes\omega_S} : S \to \mathbb{P}^n$ is called the *adjunction mapping of* $\mathscr{L}$ *on* $S$. One problem with studying $(S, \mathscr{L} \otimes \omega_S)$ is that $\mathscr{L} \otimes \omega_S$ is not generated by global sections. Fortunately, adjunction theory was revived in the 1980's and the revival led Sommese [Som79] and Van de Ven [VdV79] to, independently, determine when $\mathscr{L} \otimes \omega_S$ is generated by global sections. We state the precise result.

**Theorem 3.** *Suppose* $\mathscr{L}$ *is very ample and* $\mathscr{L} \otimes \omega_S$ *is not generated by global sections. Then exactly one of the following is true:*

*(1).* $(S, \mathscr{L}) \simeq (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(t))$, *where* $t = 1$ *or* $t = 2$.
*(2).* $(S, \mathscr{L}) \simeq (Q, \mathscr{O}_Q(1))$, *where* $Q$ *is the smooth quadric surface* $\subset \mathbb{P}^3$.
*(3).* $(S, \mathscr{L}) \simeq (\mathbb{F}_1, \mathscr{O}_{\mathbb{F}_1}(1))$, *where* $\mathbb{F}_1 = \mathbb{P}^1(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1})$.

Note that Theorem 3 asserts that the associated projective map $\varphi_{\mathscr{L}\otimes\omega_S}$ of $\mathscr{L} \otimes \omega_S$ is always a morphism, except for four particular cases. But this assertion does not give any information on the dimension of $\varphi_{\mathscr{L}\otimes\omega_S}(S)$. Sommese [Som80] has solved this by giving possibilities for $\varphi_{\mathscr{L}\otimes\omega_S}(S)$ in the next theorem.

**Theorem 4.** *Suppose* $\mathscr{L}$ *is very ample and* $\mathscr{L} \otimes \omega_S$ *is generated by global sections. Then exactly one of the following is true:*

*(1).* $\varphi_{\mathscr{L}\otimes\omega_S}(S) = \{\mathrm{pt}\}$, *and* $S$ *is a Del Pezzo surface.*
*(2).* $\varphi_{\mathscr{L}\otimes\omega_S}(S)$ *is a curve, and* $S$ *is a ruled surface over conics.*
*(3).* $\varphi_{\mathscr{L}\otimes\omega_S}(S)$ *is a surface, and* $\deg \varphi_{\mathscr{L}\otimes\omega_S} \in \{1, 2\}$.

In the case of Theorem 4.3, Sommese and Van de Ven [SV87] have described all possibilities for $\mathscr{L}$ when $\deg \varphi_{\mathscr{L}\otimes\omega_S} = 2$. The exact result contains one more possibility then we will be stating. Since we will be considering rational surfaces in this thesis, we rule out the non-rational case and state the following theorem.

**Theorem 5.** *Suppose $\mathscr{L}$ is very ample, $\mathscr{L} \otimes \omega_S$ is big and nef, and* $\deg \varphi_{\mathscr{L}\otimes\omega_S} = 2$. *Then $(S, \mathscr{L})$ is one of the following:*

*(1).* $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_7)$, *and* $\mathscr{L} \simeq \mathscr{O}_S(6L - \sum_{i=1}^{7} 2E_i)$.

*(2).* $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_8)$, *and* $\mathscr{L} \simeq \mathscr{O}_S(6L - \sum_{i=1}^{7} 2E_i - E_8)$.

*(3).* $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_8)$, *and* $\mathscr{L} \simeq \mathscr{O}_S(9L - \sum_{i=1}^{8} 3E_i)$.

## 3.2 Numerical invariants.

From now on $S$ will always denote a linearly normal smooth rational surface in $\mathbb{P}^5$. In [Gro93], Gross has classified smooth surfaces contained in quadrics by considering congruences of lines in $\mathbb{G}(1, \mathbb{P}^3)$. Therefore, we wish to consider smooth linearly rational surfaces in $\mathbb{P}^5$ *not* contained within quadrics. This consideration is our reason for the choice of $\deg S$ and $\pi_S$ as we shall see from the following proposition.

**Proposition 6.** *Let $H$ be a very ample divisor on $S$ associated to an embedding $S \hookrightarrow \mathbb{P}^5$. Suppose $H$ is not contained within a quadric. Then $\pi_S \leq 2(\deg S - 7)$.*

*Proof.* Suppose $h^0(\mathscr{I}_H(2)) = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_H \longrightarrow \mathscr{O}_{\mathbb{P}^4} \longrightarrow \mathscr{O}_H \longrightarrow 0.$$

Twisting by 2 and taking cohomology we obtain $h^0(\mathscr{O}_H(2)) \geq h^0(\mathscr{O}_{\mathbb{P}^4}(2))$. Combining the latter inequality with $h^0(\mathscr{O}_H(2H)) \geq h^0(\mathscr{O}_H(2))$ and applying Riemann-Roch we get $2H^2 + 1 - \pi_H \geq \binom{4+2}{2} = 15$, since $h^1(\mathscr{O}_H(2H)) = 0$. Rearranging, we obtain the desired inequality $\pi_H \leq 2(H^2 - 7)$. □

Due to Ionescu's classification of smooth projective varieties of degree $\leq 7$ in [Ion82], Gross [Gro93], Arrondo and Sol's [AS89] work on surfaces of degree $\leq 10$ we will be considering surfaces of degree $\deg S = 11$. Note that if $\deg S = 11$, then equality occurs in Proposition 6 if and only if $\pi_S = 8$. These invariants will be our choice.

One of the main differences between surfaces in $\mathbb{P}^4$ and surfaces in $\mathbb{P}^5$ is that every surface can be embedded into $\mathbb{P}^5$ through generic projection. See Proposition IV.5 in [Bea96] for a proof. In particular, this implies that we cannot expect a relation between invariants of $S \subset \mathbb{P}^5$, similar to the double-point formula, see Example A4.1.3 in [Har77], of $\mathbb{P}^4$ which states that $c_{\mathrm{codim}(S,\mathbb{P}^4)}(\mathscr{N}_{S|\mathbb{P}^4}) - (\deg S)^2 = 0$ when $S \subset \mathbb{P}^4$. The double-point formula for surfaces in $\mathbb{P}^4$ plays an important role since it completely determines $K_S^2$ when we are given $\deg S$ and $\pi_S$. Since we will be working with surfaces in $\mathbb{P}^5$, our strategy will be to limit ourselves to finitely many possibilities for $K_S^2$ when we are given $\deg S$ and $\pi_S$. Before doing so we need some notation.

Let $S_i$ be a surface and let $\varphi_i : S_i \hookrightarrow \mathbb{P}^N$ be an embedding such that $\mathscr{O}_S(H_i) \simeq \varphi_i^* \mathscr{O}_{\mathbb{P}^N}(1)$. Denote $H_{i+1}$ as the adjoint divisor of $H_i$, that is $H_{i+1} := H_i + K_i$ where $K_i := K_{S_i}$. Furthermore, let $\varphi_{i+1}$ denote the adjunction mapping of $H_i$ on $S_i$, let $S_{i+1} := \varphi_{i+1}(S_i)$ and let $\pi_i := \pi_{H_i}$. We say that $\varphi_{i+1}$ is the $(i+1)$-th adjunction map of $H_0 := H$ on

$S_0 := S$. We are now prepared to find finitely many choices for $K_S^2$ and to prove relations between invariants obtained by iterating the adjunction mapping.

**Proposition 7.** *Let $r_i := 9 - K_{S_i}^2$. Suppose $H_i$ is very ample. Then:*

*(1).* $\varphi_{i+1}$ *maps* $S_i$ *into* $\mathbb{P}^{\pi_i - 1}$.

*(2).* $\lceil 9 - \frac{(H_i.K_i)^2}{H_i^2} \rceil \leq r_i \leq 11 + H_i.(H_{i+1} + K_i) - \pi_i$.

*(3).* $H_{i+1}.K_{i+1} = H_{i+1}.K_i$

*(4).* $\pi_{i+1} = \pi_i + H_{i+1}.K_i$

*(5).* *If* $H_i.K_i \geq -2$, *then* $K_i^2 < 0$.

*Proof. (1).* Combining Riemann-Roch and the adjunction formula yields $\chi(S_i, \mathscr{O}(H_{i+1})) = \pi_i$. Since $S_i$ is smooth and $H_i$ is very ample, Kodaira vanishing theorem implies that $H^1(S_i, \mathscr{O}(H_{i+1})) = 0$ and the rationality of $S_i$ implies that $H^2(S_i, \mathscr{O}(H_{i+1})) = 0$. Hence $h^0(S_i, \mathscr{O}(H_{i+1}) = \pi_i$.
*(2).* The Hogde index inequality yields $9 - r_i < \frac{(H_i.K_i)^2}{H_i^2}$ which in turn gives the wanted lower bound for $r_i$. Non-degeneracy of $S_i$ implies that $1 + \operatorname{codim}(S_i, \mathbb{P}^{\pi_i - 1}) = \pi_i - 2 \leq (H_i + K_i)^2 = H_i.(H_i + 2K_i) + 9 - r_i$, by Proposition 1 and Proposition 7.1. Now, rearrange to get upper bound for $r_i$.
*(3).* Let $H_i \equiv aL - \sum_{j=1}^{r_i} b_j E_j$, where $a > 0$ and $b_j \geq 0$. Then $H_i + K_i \equiv (a - 3)L - \sum_{\{j \mid b_j \geq 1\}} (b_j - 1)E_j$. The equality follows by taking intersection products with $K_i \equiv -3L + \sum_{j=1}^{r_i} E_j$ and $K_{i+1} \equiv -3L + \sum_{\{r_i \geq j \geq 1 \mid b_j \geq 1\}} E_j$.
*(4).* Adjunction formula together with Proposition 7.3 gives us $\pi_{i+1} = \frac{1}{2}(2H_{i+1}.K_i + H_i.(H_i + K_i)) + 1$. Another use of adjunction formula gives us $H_i.(H_i + K_i) = 2\pi_i - 2$.
*(5).* See Lemma 8.3 in [Ran88]. □

Note that Proposition 7.2 gives us finitely many choices for $K_S^2$. It should also be noted that when $\deg S = 11$ and $\pi_S = 8$, then our upper bound in Proposition 7.2 is equally sharp as the double point inequality, Lemma 8.2.1 in [BS95]. The idea now is to study the adjunction mapping and the adjoint divisor of $H$ on $S$, for each choice of $K_S^2$. Using the results in Section 3.1 we will then be able to reconstruct possibilities for $H$. We make this idea more precise.

Suppose we are given the degree $H_i^2$ of a very ample divisor associated to the projective embedding of a surface $S_i$ and suppose we are also given the sectional genus $\pi_i$ of $S_i$. We wish to describe all explicit possibilities for $H_i$. Using the adjunction formula, we can determine $H_i.K_i$. Then we can use Proposition 7.2, and possibly Proposition 7.5, to find $m_i, M_i \in \mathbb{Z}$ such that $m_i \leq r_i \leq M_i$, where $K_i^2 = 9 - r_i$. For each choice of $r_i$, we can compute the invariants $\pi_{i+1}$ and $H_{i+1}^2$ by Proposition 7.4 and Proposition 7.3. Then we can consider the adjunction mapping of $H_i$, namely $\varphi_{i+1}$. By Proposition 7.1, $\varphi_{i+1} : S_i \to \mathbb{P}^{\pi_i - 1}$. Checking whether $(S_i, \mathscr{O}_{S_i}(H_i))$ is very ample, using Theorem 3 to discard the exceptional cases, we can assume $S_{i+1} = \varphi_{i+1}(S_i) \subset \mathbb{P}^{\pi_i - 1}$. Then Theorem 4 tells that not only is $0 \leq \dim S_{i+1} \leq 2$ but the theorem also describes $H_{i+1}$ in the two cases $0 \leq \dim S_{i+1} \leq 1$. These descriptions can be used to reconstruct possibilities for

$H_i$. If $\varphi_{i+1}(S_i)$ is a surface, then we can use Theorem 5 to reconstruct the possibilities for $H_i$, when $\deg \varphi_{i+1} = 2$. This leaves us with the case when $\varphi_{i+1}(S_i)$ is a surface and $\deg \varphi_{i+1} = 1$. For a reconstruction of $H_i$, in the latter case, we will depend upon classifications of smooth linearly normal rational surfaces $\mathbb{P}^N$, where $N \leq 4$. In other words, we will check whether $\pi_i \leq 5$ or not. If $\pi_i > 5$, then we shall apply the same procedure with $H_{i+1}$ instead, i.e. study the adjoint divisor of $H_{i+1}$ on $S_{i+1}$ instead. We summarize this procedure in an algorithm.

**Algorithm 8.**

```
Input:  $H_i^2$ and $\pi_i$.
Compute:  $\pi_{i+1}$, $H_{i+1}.K_{i+1}$ and $m_i, M_i \in \mathbb{Z}$ s.t.  $m_i \leq r_i \leq M_i$.
For $r_i = m_i \to M_i$:
    Case $\dim \varphi_{i+1}(S_i) = 0$:  Use Theorem 4.1.
    Case $\dim \varphi_{i+1}(S_i) = 1$:  Use Theorem 4.2.
    Case $\dim \varphi_{i+1}(S_i) = 2$:
        Case $\deg \varphi_{i+1}(S_i) = 2$:  Use Theorem 5.
        Case $\deg \varphi_{i+1}(S_i) = 1$:
            If $\pi_i \leq 5$:  Use earlier classifications.
            If $\pi_i > 5$:  Run algorithm with input:  $H_{i+1}^2$ and $\pi_{i+1}$.
End.
Output:  Explicit descriptions of $H_i$.
```

Note that when we are applying the algorithm above we are assuming $H_i$ actually defines an embedding of $S_i \hookrightarrow \mathbb{P}^5$. This means that if some description of $H_i$ provides an embedding of $S_i \hookrightarrow \mathbb{P}^5$, then that description is neccesarily found in the output of Algorithm 8. In section 3.4 and in Chapter 4, we will deal with the weeding out of the false descriptions from the true descriptions. From now on, by *adjunction process* we shall mean Algorithm 8.

A natural question to ask is whether the adjunction process terminates or not, given the inputs $H_0^2 = 11$ and $\pi_0 = 8$. We answer not only this question, but we also determine the largest $i$ such that the $i$-th adjunction mapping that has to be considered when the adjunction process is applied with the inputs above. At this point the reader may wish to have a look at Appendix B.

**Lemma 9.** *The adjunction process terminates when the inputs are* $\deg S = 11$ *and* $\pi_S = 8$. *Moreover* $-11 \leq K_S^2 \leq -1$. *In particular, termination is executed in the 7th adjunction mapping and:*

*(1). If* $-11 \leq K_S^2 \leq -6$, *then* $\pi_1 \leq 5$ *s.t. termination is executed in 2nd adj. map.*
*(2). If* $-5 \leq K_S^2 \leq -4$, *then* $\pi_2 \leq 5$ *s.t. termination is executed in 3rd adj. map.*
*(3). If* $K_S^2 = -3$, *then* $\pi_3 \leq 5$ *s.t. termination is executed in 4th adj. map.*
*(4). If* $K_S^2 = -2$, *then* $\pi_5 \leq 5$ *s.t. termination is executed in 6th adj. map.*
*(5). If* $K_S^2 = -1$, *then* $\pi_6 \leq 5$ *s.t. termination is executed in 7th adj. map.*

*Proof.* The idea is to simply check when $\pi_i \leq 5$. Using Proposition 7.2, we have $-11 \leq K_S^2 \leq -1$. In fact, this gives us $10 \leq r_0 \leq 20$. Since $\pi_0 = 8 > 5$ we check $\pi_1$ instead. Now, $\pi_1 = 20 - r_0 \leq 5$ if and only if $15 \leq r_0 \leq 20$. Suppose $10 \leq r_0 \leq 14$. We check $\pi_2$. Then Proposition 7.5 applies since $H_1.K_1 = 12 - r_0 \geq -2$, giving us $10 \leq r_2$. Now, $\pi_2 = 41 - 2r_0 - r_1 \leq 5$ whenever $r_1 \geq 36 - 2r_0$, which is satisfied for $13 \leq r_0 \leq 14$ due to $r_1 \geq 10$. Suppose $10 \leq r_0 \leq 12$. We consider each choice for $r_0$. Suppose $r_0 = 12$. Then $10 \leq r_1 \leq 12$ and by Proposition 7.2 we obtain $9 \leq r_2 \leq r_1$. We check $\pi_3$. Now, $\pi_3 = 35 - 2r_1 - r_2 \leq 5$ for every combination of $(r_0, r_1, r_2)$ except $(12, 10, 9)$. But the combination $(12, 10, 9)$ cannot occur since $H_2.K_2 = -1 > -2$ yields $r_2 > 9$. Suppose $r_0 = 11$. Then $H_1.K_1 = 0 > -2$ and $H_2.K_2 = -1 > -2$ since $r_1 \leq r_0 = 11$. So we have $10 \leq r_1 \leq 11$ and $10 \leq r_2 \leq r_1$, i.e. $(r_0, r_1, r_2) \in \{(11, 11, 11), (11, 11, 10), (11, 10, 10)\}$. We check each combination. In case $r_1 = r_2 = 11$, then $\pi_3 = 5$. In both cases $r_1 = r_2 + 1 = 11$ and $r_1 = r_2 = 10$, we have $H_3.K_3 \geq -2$ which gives us $r_3 = 10$ for both cases, i.e. $(r_0, r_1, r_2, r_3) \in \{(11, 11, 10, 10), (11, 10, 10, 10)\}$. The case $(11, 11, 10, 10)$ yields $\pi_4 = 3 < 5$. For the case $(11, 10, 10, 10)$, using Proposition 7.2 we get $\lceil 9 - \frac{(H_4.K_4)^2}{H_4^2} \rceil = 9$ which gives $r_4 \geq 9$. Combining the latter inequality with $r_4 \leq r_3 = 10$ implies $\pi_5 < 5$. Finally, we are left with the last case $r_0 = 10$ in which case $H_j.K_j \geq -2$ for $j \leq 5$. So the only combination for $(r_0, ..., r_5)$ is $(10, ..., 10)$, in which case $\pi_6 = 5$. $\square$

### 3.3 Explicit linear systems.

In this section $S_0 := S$ will always denote a linearly normal smooth rational surface of degree 11 and sectional genus 8. Note that by Lemma 9, we only need to consider the $i$-th adjunction map $\varphi_i$ where $1 \leq i \leq 7$. At this point the reader may wish to have a look at Appendix A.

First, we describe the possibilities for $H_0$ when $\varphi_i(S_{i-1})$ is a point.

**Proposition 10.** *Suppose that* $\varphi_i(S_{i-1})$ *is a point. Then the following is true:*
$K_S^2 = -10$, $i = 2$, *and* $H \equiv 6L - \sum_{i=1}^{2} 2E_i - \sum_{j=3}^{19} E_j$.

*Proof.* Let $1 \leq i \leq 7$. Suppose $\dim \varphi_i(S_{i-1}) = 0$. Then Theorem 4.1 tells us that $S_{i-1}$ is a Del Pezzo surface, that is $H_{i-1} \equiv -K_{i-1}$ or equivalently $H_i \equiv 0$. The idea now is to exploit the numerical equivalence of $H_i$ to the zero divisor. Note that we have:

$$(1).\ H_0^2 = (-\sum_{j=0}^{i-1} K_j)^2 = 11,\ (2).\ H_{i-1}.H_i = 0,\ (3).\ H_i^2 = 0 \text{ and } (4).\ \pi_0 = 8.$$

If $i=1$, then (1) reduces to $9-r_0=11$ which is never possible for $r_0 \geq 0$. If $i=2$, then (2) and (3) give us $38-2r_0=0$ and $59-3r_0-r_1=0$, respectively. Combining the latter equations we get $r_0=19$ and $r_1=2$. If $i=3$, then (2) and (4) give us $40-2r_0-r_1=0$ and $28-r_1-2r_2=8$, respectively. Combining the latter relations we obtain $r_0=r_2+10$ and $r_1=2(r_2-10)$. By Lemma 7 we may assume $10 \leq r_0 \leq 14$ in which case $H_1.K_1 \geq -2$ implies $r_1 \geq 10$, that is $r_2 \geq 15$. But then $r_2 > r_0$ contradicts $r_2 \leq r_1 \leq r_0$. For $4 \leq i \leq 7$, we will consider them as one case. Writing out (1) explicitly $H_i^2 = \alpha(i) - \sum_{j=0}^{i-1}(2(i-j)-1)r_j$, where $\alpha(i) = H_0^2 + 2iH_0.K_0 + \sum_{j=0}^{i-1} 9(2j+1) = 11+3i(2+3i)$. Furthermore, since $H_0 \equiv -\sum_{j=0}^{i-1} K_j$ we have $\pi_0 = \beta(i) - \sum_{j=1}^{i-1} jr_j$, where $\beta(i) = \binom{3i-1}{2}$. Then $H_i^2 - (2(i-1)-1)\pi_0 = \alpha(i) - (2i-3)\beta(i) - (2i-1)r_0 - \sum_{j=1}^{i-1}[2i+j-2ij-1]r_j$. Rearranging and using (3) and (4), we get

$$(5).\ r_0 = \frac{2i-3}{2i-1}(8-\beta(i)) + \frac{1}{2i-1}\alpha(i) + \sum_{j=1}^{i-1}(j-1)r_j.$$

If $i=4$, then (5) gives us $r_0 = r_2+2r_3-8$. By Lemma 9 we may assume $10 \leq r_0 \leq 12$, in which case $10 \leq r_j \leq r_{j-1}$ since $H_j.K_j \geq -2$ for $j \leq 3$. This gives us $r_0 \geq 22$, which contradicts $r_0 \leq 12$. If $i=5$, then (5) gives us $r_0 = r_2+2r_3+2r_4-35$. By Lemma 9 we may assume $10 \leq r_0 \leq 12$, in which case $10 \leq r_j \leq r_{j-1}$ since $H_j.K_j \geq -2$ for $j \leq 4$. This gives us $r_0 \geq 15$, which contradicts $r_0 \leq 12$. If $i=6$, then (5) gives us $r_0 = r_5+2r_4+3r_3+4r_2-71$. By Lemma 9 we may assume $r_0=10$, in which case $10 \leq r_j \leq r_{j-1}$ since $H_j.K_j \geq -2$ for $j \leq 4$. This gives us $r_0 \geq 19+r_5$, which contradicts $r_0 = 10$. $\square$

Second we describe the possibilities for $H_0$ when $\varphi_i(S_{i-1})$ is a curve.

**Proposition 11.** *Let $r_i = 9-K_{S_i}^2$. Suppose that $\varphi_i(S_{i-1})$ is a curve. Then $H$ is in the following form: $H \equiv 2iB + (\alpha+2(i-1)-ie) - \sum_{j=1}^{r_{i-1}-1} iE_j - \sum_{\beta=1}^{i-1}\sum_{\gamma=r_\beta+1}^{r_{\beta-1}} \beta E_\gamma$, where $e \leq \alpha = \frac{1}{3}(\pi_{i-1}+r_{i-1}-4)$.*

*Proof.* Let $1 \leq i \leq 7$. Suppose $\dim\varphi_i(S_{i-1}) = 1$. Then Theorem 4.2 tells us that $S_{i-1}$ is a ruled surface over conics, that is $H_{i-1} \equiv 2B + (\alpha-e)F - \sum_{j=1}^{r_{i-1}-1} E_j$ since there are $r_{i-1}-1$ singular fibres. Furthermore, since $K_{i-1} \equiv -2B-(2+e)F$ we have $H_0 \equiv 2iB + (\alpha+2(i-1)-ie) - \sum_{j=1}^{r_{i-1}-1} iE_j - \sum_{\beta=1}^{i-1}\sum_{\gamma=r_\beta+1}^{r_{\beta-1}} \beta E_\gamma$. To estimate $\alpha$, we use $\pi_{i-1} = h^0(\mathscr{O}_{S_{i-1}}(H_{i-1})) = h^0(\mathrm{Sym}^2(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(e)) \otimes \mathscr{O}_{\mathbb{P}^1}(\alpha-e)) - r_{i-1}+1 = 3(1+\alpha)+1-r_{i-1}$. That is, $\alpha = \frac{1}{3}(\pi_{i-1}+r_{i-1}-4)$. $\square$

Next, we determine the possibilities for $\varphi_i(S_{i-1})$ having degree 2.

**Proposition 12.** *There are no possibilities for $H$, when $\dim\varphi_i(S_{i-1}) = 2$, $\deg\varphi_i = 2$ and $p_a(-\sum_{j=0}^{i-1} K_j) \geq 0$.*

*Proof.* Let $D = 3L - \sum_{j=1}^{7} E_j$ and let $1 \leq i \leq 7$. Suppose $\dim\varphi_i(S_{i-1}) = 2$ and suppose $\deg\varphi_i = 2$. Then Theorem 5 tells us that $H_{i-1}$ is one of the following:

$$(1).\ H_{i-1} \equiv 2D,\ (2).\ H_{i-1} \equiv 2D - E_8 \text{ or } (3).\ H_{i-1} \equiv 3D - 3E_8.$$

This means that $H_0 \equiv A - \sum_{j=0}^{i-1} K_j$, where $A \equiv H_{i-1}$ is one of the three cases above. The idea now is that since $p_a(-\sum_{j=0}^{i-1} K_j) \geq 0$, we have

$$(4).\ p_a(A) + \sum_{j=0}^{i-1} A.(-K_j) - 1 \leq p_a(H_0).$$

For case (1), $A \equiv 2D$ where $p_a(2D) = 3$ and $2D.(-K_j) = 18 - 2\min\{7, r_j\}$. Using the latter relations with (4) we get $8 \geq 2 + 18i - 2\sum_{j=0}^{i-1} \min\{7, r_j\} \geq 2 + 18i - 14i$, that is $i \leq \frac{6}{4}$. So it suffices to check (1) for $i = 1$. Now, case (1) and $i = 1$ is not possible since $H_0^2 = 4D^2 \neq 11$. For case (2), $A \equiv 2D - E_8$ where $p_a(2D - E_8) = 3$ and $(2D - E_8).(-K_j) = 18 - 2\min\{7, r_j\} - \epsilon_j$, where $\epsilon_j = 1$ if $r_j \geq 8$ and $\epsilon_j = 0$ if $r_j < 8$. Using the latter relations with (4) we get $8 \geq 2 + 18i - 2\sum_{j=0}^{i-1} \min\{7, r_j\} - \sum_{j=0}^{i-1} \epsilon_j \geq 2 + 18i - 14i - i$, that is $i \leq \frac{6}{3} = 2$. So it suffices to check (2) for $1 \leq i \leq 2$. Now, case (2) and $i = 1$ is not possible since $H_0^2 = 4D^2 - 1 \neq 11$. Case (2) and $i = 2$ is not possible since $H_0 \equiv 2D - E_8 - K_0 \equiv 9l - \sum_{i=1}^{7} 3E_i - 2E_8 - \sum_{j=9}^{r_0} E_j$, since $r_0 > 8$, has sectional genus $\pi_0 = \binom{8}{2} - 7\binom{3}{2} - \binom{2}{2} \neq 8$. For case (3), $A \equiv 3D - 3E_8$ where $p_a(3D - 3E_8) = 4$ and $(3D - 3E_8).(-K_j) = 27 - 3\min\{8, r_j\}$. Using the latter relations with (4) we get $8 \geq 3 + 27i - 2\sum_{j=0}^{i-1} \min\{8, r_j\} \geq 3 + 27i - 24i$, that is $i \leq \frac{5}{3}$. So it suffices to check (3) for $i = 1$. Now, case (3) and $i = 1$ is not possible since $H_0^2 = 3^2(D - E_8)^2 \neq 11$. ☐

Before going any further we consider the case the case $K_S^2 = -11$, which is the only case we don't need to go further then the first adjunction mapping.

**Proposition 13.** *There are no possibilities for $H$ when $K_S^2 = -11$.*

*Proof.* Suppose $r_0 = 20$. The first adjunction map $\varphi_1$ maps $S_0$ into $\mathbb{P}^7$ and $S_1 = \varphi_1(S_0)$ is a surface of degree 6. Therefore, $S_1 \subset \mathbb{P}^7$ is a surface of minimal degree. Then Theorem 2 tells us that $S_1$ is either a Veronese surface or a rational normal scroll. If $S_1$ is a Veronese surface, then $H_1 \equiv 2L$ gives us $H_0 \equiv 5L - \sum_{i=1}^{20} E_i$ which has degree $H_0^2 = 5 \neq 11$. So $S_1$ must be a rational normal scroll, in which case $H_1 \equiv B + (\alpha - e)F$ gives us $H_0 \equiv 3B + (\alpha + 2 - 2e)F - \sum_{i=1}^{20} E_i$. To determine $\alpha$, recall that every surface of minimal degree $d$ satisfies $d = 2\alpha - e$ by Corollary IV.2.19 in [Har77]. In our case, $6 = 2\alpha - e$ is satisfied for $0 \leq e < \alpha$ if and only if $(\alpha, e) \in \{(3, 0), (4, 2), (5, 4)\}$. On the other hand, the degree $H_0^2 = 11$ if and only if $(6 + e)(\alpha + 2 - 2e) = 31$ if and only if 31 is not a prime integer or not a positive integer. ☐

Now we shift our attention towards the cases $-10 \leq K_S^2 \leq -6$. These are exactly the cases we have to consider the second adjunction mapping. Note that in case $K_S^2 = -10$, then $\varphi_2(S_1) \subset \mathbb{P}^0 = \{\text{pt}\}$ and so the only possibility for $H$ is the conclusion of Proposition 10. Therefore, we prove the following result for $-9 \leq K_S^2 \leq -6$.

**Proposition 14.** *Suppose* $-9 \leq K_S^2 \leq -6$*. Then* $H$ *is one of the following:*

*(1).* $K_S^2 = -8$ *and* $H \equiv 7L - \sum_{i=1}^{7} 2E_i - \sum_{j=8}^{17} E_j$*.*

*(2).* $K_S^2 = -7$ *and* $H \equiv 5B + 5F - \sum_{i=1}^{9} 2E_i - \sum_{j=10}^{16} E_j$*, where* $e = 0$*.*

*(3).* $K_S^2 = -7$ *and* $H \equiv 9L - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{8} 2E_j - \sum_{k=9}^{16} E_k$*.*

*(4).* $K_S^2 = -6$ *and* $H \equiv 8L - 3E_1 - \sum_{i=2}^{11} 2E_i - \sum_{j=12}^{15} E_j$*.*

*(5).* $K_S^2 = -6$ *and* $H \equiv 9L - \sum_{i=1}^{5} 3E_i - \sum_{j=6}^{10} 2E_j - \sum_{k=11}^{15} E_k$*.*

*(6).* $K_S^2 = -6$ *and* $H \equiv 10L - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{15} E_j$*.*

*Proof.* The second adjunction mapping $\varphi_2$ maps $S_1$ into $\mathbb{P}^{19-r_0}$ and $S_2 = \varphi_2(S_1)$ has degree $59 - 3r_0 - r_1$. If $r_0 = 18$, then $S_2 \subset \mathbb{P}^1$ and $S_2$ may either be a point or a curve. The case of $S_2$ being a point is covered by 10. In the case $S_2$ is a curve we can use Proposition 11. If $r_0 = 17$, then $S_2 \subset \mathbb{P}^2$. Proposition 10 and Proposition 11 takes care of $\dim S_2 < 2$ cases. So we may assume $S_2$ is a surface, in which case $S_2 \simeq \mathbb{P}^2$ gives us $H_2 \equiv L$ since $\deg \mathbb{P}^2 = 1$. Then $H_0 \equiv 7L - \sum_{i=1}^{r_1} 2E_i - \sum_{j=r_1+1}^{17} E_j$. Furthermore, $\pi_0 = \binom{6}{2} - r_0\binom{2}{2} = 8$ gives $r_1 = 7$ in which case $H_0^2 = 11$. If $r_0 = 16$, then $S_2 \subset \mathbb{P}^3$. Proposition 10 and Proposition 11 takes care of $\dim S_2 < 2$ cases. So $S_2$ is a surface of degree $11 - r_1$. Furthermore, $8 \leq r_1 \leq 9$ since $r_0 = 16$. Therefore, $S_2$ is either the quadric or the cubic surface in $\mathbb{P}^3$. If $r_1 = 9$, then $S_2 \simeq \mathbb{F}_0$ and therefore $H_2 \equiv B + F$, which gives us $H_0 \equiv 5B + 5F - \sum_{i=1}^{9} 2E_i - \sum_{j=10}^{15} E_j$. If $r_1 = 8$, then $H_2 \equiv 3L - \sum_{i=1}^{6} E_i$ gives us $H_0 \equiv 9L - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{8} 2E_j - \sum_{k=9}^{16} E_k$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_0 = 15$, then $S_2 \subset \mathbb{P}^4$. Proposition 10 and Proposition 11 takes care of $\dim S_2 < 2$ cases. So $S_2$ is a surface of degree $14 - r_1$. Furthermore, $9 \leq r_1 \leq 11$. If $r_1 = 11$, then $S_2 \subset \mathbb{P}^4$ has degree 3 and so $H_2 \equiv 2L - E_1$, by using Appendix A. This gives us $H_0 \equiv 8L - 3E_1 - \sum_{i=2}^{11} 2E_i - \sum_{j=12}^{15} E_j$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_1 = 10$, then $S_2 \subset \mathbb{P}^4$ has degree 4 and so $H_2 \equiv 3L - \sum_{i=1}^{5} E_i$, by using Appendix A. This gives us $H_0 \equiv 9L - \sum_{i=1}^{5} 3E_i - \sum_{j=6}^{10} 2E_j - \sum_{k=11}^{15} E_k$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_1 = 9$, then $S_2 \subset \mathbb{P}^4$ has degree 5 and so $H_2 \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_i$. This gives us $H_0 \equiv 10L - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{15} E_j$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. $\square$

Next we study the cases $-5 \leq K_S^2 \leq -4$. By Lemma 9, these are exactly the cases where we don't need to go beyond the third adjunction mapping.

**Proposition 15.** *Suppose* $K_S^2 = -5$*. Then* $H$ *is one of the following:*

*(1).* $H \equiv 8L - \sum_{i=1}^{13} 2E_i - E_{14}$*.*

*(2).* $H \equiv 10L - \sum_{i=1}^{9} 3E_i - 2E_{10} - \sum_{k=11}^{14} E_k$*.*

*Proof.* Suppose $r_0 = 14$. Then the third adjunction mapping $\varphi_3$ maps $S_2$ into $\mathbb{P}^{12-r_1}$ and $S_3 = \varphi_3(S_2)$ has degree $40 - 3r_1 - r_2$. Furthermore, $9 \leq r_1 \leq 13$. If $r_1 = 13$, then we look at the second adjunction mapping instead, since then $S_2 \subset \mathbb{P}^5$ is a surface of degree 4. By Theorem 2, $S_2$ is then either a Veronese surface of a rational normal scroll. In the case of a Veronese surface, $H_2 \equiv 2L$ gives us $H_0 \equiv 8L - \sum_{i=1}^{13} 2E_i - E_{14}$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $S_2$ is a rational normal scroll, then $H_2 \equiv B + (\alpha - e)F$ gives us $H_0 \equiv 5B + (\alpha + 4 - 3e)F - \sum_{i=1}^{13} 2E_i - E_{14}$. To estimate $\alpha$, we recall that

$4 = 2\alpha - e$ is satisfied for $(\alpha, e) \in \{(3,2),(2,0)\}$. Neither case of $(\alpha, e)$ yields $H_0^2 = 11$. For $9 \le r_1 \le 12$, we return to the third adjunction map. If $r_1 = 12$, then $S_3 \subset \mathbb{P}^0$ and Proposition 10 gives no possibilities. If $r_1 = 11$, then $S_3 \subset \mathbb{P}^1$ then Proposition 10 and Proposition 11 take care of this case. If $r_1 = 10$, then $S_3 \subset \mathbb{P}^2$. Proposition 10 and Proposition 11 covers the cases $\dim S_3 < 2$. So if $S_3$ is a surface, then $S_3 \simeq \mathbb{P}^2$ which gives us $H_3 \equiv L$ such that $H_0 \equiv 10L - \sum_{i=1}^{r_2} 3E_i - \sum_{j=r_2+1}^{10} 2E_j - \sum_{k=11}^{14} E_k$. Then $\pi_0 = 26 - 2r_2 = 8$ gives $r_2 = 9$, in which case $H_0^2 = 11$ also. If $r_1 = 9$, then $S_3 \subset \mathbb{P}^3$ has degree $13 - r_2$. Furthermore, $9 \le r_2$ combined with $r_2 \le r_1 \le 9$ gives us $r_2 = 9$. Then $S_3 \subset \mathbb{P}^3$ has degree 4, but there are no rational surfaces in $\mathbb{P}^3$ of degree 4. $\square$

**Proposition 16.** *Suppose $K_S^2 = -4$. Then $H$ is one of the following:*

*(1).* $H \equiv 10L - \sum_{i=1}^{8} 3E_i - \sum_{j=9}^{12} 2E_j - E_{13}$.

*(2).* $H \equiv 7B + 7F - \sum_{i=1}^{10} 3E_i - 2E_{11} - \sum_{j=12}^{13} E_j,$ *where* $e = 0$.

*(3).* $H \equiv 12L - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{9} 3E_j - \sum_{k=10}^{11} 2E_k - \sum_{t=12}^{13} E_t$.

*Proof.* Suppose $r_0 = 13$. Then the third adjunction mapping $\varphi_3$ maps $S_2$ into $\mathbb{P}^{14-r_1}$ and $S_3 = \varphi_3(S_2)$ has degree $45 - 3r_1 - r_2$. Furthermore, $9 \le r_1 \le 15$. But as $r_1 \le r_0$, we instead get $9 \le r_1 \le 13$. Using Proposition 7.5, since $H_1.K_1 = -1 > -2$, we get $r_1 \ge 10$. If $r_1 = 13$, then $S_3 \subset \mathbb{P}^1$ such that Proposition 10 and Proposition 11 finishes this case. If $r_1 = 12$, then $S_3 \subset \mathbb{P}^2$. The cases $\dim S_3 < 2$ are taken care of by Proposition 10 and Proposition 11. So if $S_3 \simeq \mathbb{P}^2$, then $H_3 \equiv L$ which gives us $H_0 \equiv 10L - \sum_{i=1}^{r_2} 3E_i - \sum_{j=r_2+1}^{12} 2E_j - E_{13}$. If $\pi_0 = 24 - 2r_2 = 8$, then must have $r_2 = 8$ which also yields $H_0^2 = 11$. If $r_1 = 11$, then $S_3 \subset \mathbb{P}^3$ has degree $12 - r_2$. Furthermore, since $S_3$ is rational we have $9 \le r_2 \le 11$. If $r_2 = 11$, then $S_3 \subset \mathbb{P}^3$ has degree 1 and so $H_3 \equiv L$ which gives $H_0 \equiv 10L - \sum_{i=1}^{11} 3E_i - \sum_{j=12}^{13} E_j$. But then $\pi_0 \ne 8$. If $r_2 = 10$, then $S_3 \subset \mathbb{P}^3$ has degree 2 and so $H_3 \equiv B + F$ which gives $H_0 \equiv 7B + 7F - \sum_{i=1}^{10} 3E_i - 2E_{11} - \sum_{j=12}^{13} E_j$, where $e = 0$. If $r_2 = 9$, then $S_3 \subset \mathbb{P}^3$ has degree 3 and so $H_3 \equiv 3L - \sum_{i=1}^{6} E_i$ which gives us $H_0 \equiv 12L - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{9} 3E_j - \sum_{k=10}^{11} 2E_k - \sum_{t=12}^{13} E_t$ which has $\pi_0 = 8$ and $H_0^2 = 11$. If $r_1 = 10$, then $S_3 \subset \mathbb{P}^4$ of degree $15 - r_2$. Furthermore, $9 \le r_2 \le r_1 \le 10$. If $r_2 = 10$, then $S_3$ has degree 5 and so $H_3 \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_i$. This gives $H_0 \equiv 13L - 5E_1 - \sum_{i=2}^{8} 4E_i - \sum_{j=9}^{10} 3E_j - \sum_{k=11}^{13} E_k$. But then $\pi_0 \ne 8$. If $r_2 = 9$, then $S_3$ has degree 6 and so $H_3 \equiv 4L - \sum_{i=1}^{10} E_i$ which gives $H_0 \equiv 13L - \sum_{i=1}^{9} 4E_i - 3E_{10} - \sum_{j=11}^{13} E_j$. But then $\pi_0 \ne 8$. $\square$

Now we study the case when the fourth adjunction mapping terminates.

**Proposition 17.** *Suppose* $\deg K_S^2 = -3$. *Then $H$ is one of the following:*

*(1).* $H \equiv 11L - 4E_1 - \sum_{i=2}^{11} 3E_i - 2E_{12}$.

*(2).* $H \equiv 12L - \sum_{i=1}^{5} 4E_i - \sum_{j=6}^{10} 3E_j - \sum_{k=11}^{12} 2E_k$.

*(3).* $H \equiv 13L - 5E_1 - \sum_{i=2}^{8} 4E_1 - 3E_9 - \sum_{j=10}^{12} 2E_j$.

*(4).* $H \equiv 13L - \sum_{i=1}^{9} 4E_1 - 3E_{10} - 2E_{11} - E_{12}$.

*(5).* $H \equiv 16L - 6E_1 - \sum_{j=2}^{8} 5E_j - \sum_{k=9}^{10} 4E_k - \sum_{t=11}^{12} E_t$.

*Proof.* Suppose $r_0 = 12$. Then $\varphi_3$ maps $S_2$ into $\mathbb{P}^{16-r_1}$ and $S_3$ has degree $50 - 3r_1 - r_2$. Furthermore, $H_1.K_1 = 0 > -2$ and $r_1 \leq r_0$ gives $10 \leq r_1 \leq 12$. If $r_1 = 12$, then $S_2 \subset \mathbb{P}^4$ is a surface of degree $14 - r_2$. Furthermore, $9 \leq r_2 \leq r_1 = 12$. In fact $r_2 \neq 12$, since if it were then $S_3 \subset \mathbb{P}^4$ would have degree 2 and be a degenerate. If $r_2 = 11$, then $S_3 \subset \mathbb{P}^4$ has degree 3 and so $H_3 \equiv 2L - E_1$ which gives $H_0 \equiv 11L - 4E_1 - \sum_{i=2}^{11} 3E_i - 2E_{12}$, which has $\pi_0 = 8$ and $H_0^2 = 11$. If $r_2 = 10$, then $S_3 \subset \mathbb{P}^4$ has degree 4 and so $H_3 \equiv 3L - \sum_{i=1}^{5} E_1$ which gives $H_0 \equiv 12L - \sum_{i=1}^{5} 4E_i - \sum_{j=6}^{10} 3E_j - \sum_{k=11}^{12} 2E_k$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_2 = 9$, then $S_3 \subset \mathbb{P}^4$ has degree 5 and so $H_3 \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_1$ which gives $H_0 \equiv 13L - 5E_1 - \sum_{i=2}^{8} 4E_1 - 3E_9 - \sum_{j=10}^{12} 2E_j$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. This leaves us with the cases $10 \leq r_1 \leq 11$. Now we move onto the fourth adjunction mapping. Then $\varphi_4$ maps $S_3$ into $\mathbb{P}^{34-2r_1-r_2}$ and $S_4$ has degree $95 - 5r_1 - 3r_2 - r_3$. If $r_1 = 11$, then $S_4 \subset \mathbb{P}^{12-r_2}$ has degree $40 - 3r_2 - r_3$. Furthermore, $H_2.K_2 \geq 6 > -2$ gives $r_2 \geq 10$ and combined with $r_2 \leq r_1$ we get $10 \leq r_2 \leq 11$. If $r_2 = 11$, then $S_4 \subset \mathbb{P}^1$ is a point or $\mathbb{P}^1$ itself, so it is taken care of by Proposition 10 and Proposition 11. If $r_2 = 10$, then $S_4 \subset \mathbb{P}^2$. The $\dim S_4 < 2$ cases are taken care of by Proposition 10 and Proposition 11. So, this leaves us with $S_4 \simeq \mathbb{P}^2$ in which case $H_4 \equiv L$ gives $H_0 \equiv 13L - \sum_{i=1}^{r_3} 4E_i - \sum_{j=r_3+1}^{10} 3E_i - 2E_{11} - E_{12}$. Then $\pi_0 = 35 - 3r_3 = 8$ yields $r_3 = 9$ in which case $H_0^2 = 11$ also. If $r_1 = 10$, then $S_4 \subset \mathbb{P}^{14-r_2}$ has degree $45 - 3r_2 - r_3$. Furthermore, $H_2.K_2 = -1 > -2$ such that $r_2 = 10$ which means that $S_4 \subset \mathbb{P}^4$ of degree $15 - r_3$. Now, $H_3.K_3 = -2$ such that $r_3 = 10$ which means that $S_4 \subset \mathbb{P}^4$ has degree 5. Then $H_4 \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_i$ which gives $H_0 \equiv 16L - 6E_1 - \sum_{j=2}^{8} 5E_j - \sum_{k=9}^{10} 4E_k - \sum_{t=11}^{12} E_t$, which has both $\pi_0 = 8$ and $H_0^2 = 11$. $\square$

Next we consider the case when the fifth and the sixth adjunction mapping terminates.

**Proposition 18.** *Suppose $K_S^2 = -2$. Then $H$ is one of the following:*

*(1).* $H \equiv 14L - 5E_1 - \sum_{i=2}^{11} 4E_i$.

*(2).* $H \equiv 15L - \sum_{i=1}^{5} 5E_i - \sum_{j=6}^{10} 4E_j - 3E_{11}$.

*(3).* $H \equiv 16L - 6E_1 - \sum_{i=2}^{8} 5E_i - 4E_9 - \sum_{j=10}^{11} 3E_j$.

*(4).* $H \equiv 16L - \sum_{i=1}^{9} 5E_i - 4E_{10} - 2E_{11}$.

*(5).* $H \equiv 19L - \sum_{i=1}^{9} 6E_i - 5E_{10} - E_{11}$.

*Proof.* Suppose $r_0 = 11$. As in the proof of Lemma 9 we have three combinations for $(r_0, r_1, r_2)$, namely $(11, 11, 11)$, $(11, 11, 10)$ and $(11, 10, 10)$. For the case $(11, 11, 11)$, we check the fourth adjunction mapping in which case $\varphi_4$ maps $S_3$ into $\mathbb{P}^4$ and $S_4$ has degree $14 - r_3$. Furthermore, $9 \leq r_3 \leq r_2 = 11$. If $r_3 = 11$, then $S_4 \subset \mathbb{P}^4$ has degree 3 and $H_4 \equiv 2L - E_1$. This means that $H_0 \equiv 14L - 5E_1 - \sum_{i=2}^{11} 4E_i$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_3 = 10$, then $S_4 \subset \mathbb{P}^4$ has degree 4 and $H_4 \equiv 3L - \sum_{i=1}^{5} E_1$. This means that $H \equiv 15L - \sum_{i=1}^{5} 5E_i - \sum_{j=6}^{10} 4E_j - 3E_{11}$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_3 = 9$, then $S_4 \subset \mathbb{P}^4$ has degree 5 and $H_4 \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_1$. This means that $H \equiv 16L - 6E_1 - \sum_{i=2}^{8} 5E_i - 4E_9 - \sum_{j=10}^{11} 3E_j$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. For the two remaining cases $(11, 11, 10)$ and $(11, 10, 10)$ we check the fifth adjunction mapping. As in the proof of Lemma 9, we have two combinations for $(r_0, r_1, r_2, r_3)$, namely $(11, 11, 10, 10)$ and $(11, 10, 10, 10)$. In case $(11, 11, 10, 10)$, $\varphi_5$ maps $S_4$ into $\mathbb{P}^2$ in which case $S_5 \simeq \mathbb{P}^2$

implies $H_5 \equiv L$. This means that $H_0 \equiv 16L - \sum_{i=1}^{r_2} 5E_i - \sum_{j=r_2+1}^{10} 4E_j - 2E_{11}$ in which case $\pi_0 = 44 - 4r_4 = 8$ yields $r_4 = 9$ and then $H_0^2 = 11$ also. In case $(11, 10, 10, 10)$, as in the proof of Lemma 9, we have two subcases $(11, 10, 10, 10, 10)$ and $(11, 10, 10, 10, 9)$. For both cases, we move onto the sixth adjunction mapping. In the case $(11, 10, 10, 10, 10)$, $\varphi_6$ maps $S_5$ into $\mathbb{P}^2$. This means that $S_6 = \varphi_6(S_5) \simeq \mathbb{P}^2$ which gives $H_6 \equiv L$. Then $H_0 \equiv 19L - \sum_{i=1}^{r_5} 6E_i - \sum_{j=r_5+1}^{10} 5E_j - E_{11}$ which has $\pi_0 = 53 - 5r_5 = 8$ which yields $r_5 = 9$. In the case $(11, 10, 10, 10, 9)$, $\varphi_6$ maps $S_5$ into $\mathbb{P}^3$ and $S_6 = \varphi_6(S_5)$ has degree $13 - r_6$. Rationality of $S_6$ implies that we must have $10 \leq r_6 \leq 12$ but this contradicts $r_6 \leq r_5 = 9$. □

Finally, we consider the case when the seventh adjunction mapping terminates.

**Proposition 19.** *Suppose $K_S^2 = -1$. Then $H$ is one of the following:*

*(1).* $H \equiv 24L - \sum_{i=1}^{5} 8E_i - \sum_{j=6}^{10} 7E_j$.

*(2).* $H \equiv 25L - 9E_1 - \sum_{i=2}^{8} 8E_i - 7E_9 - 6E_{10}$.

*Proof.* Suppose $r_0 = 10$. As in the proof of Lemma 9, there is only one combination for $(r_1, ..., r_5)$, namely $(10, ..., 10)$. In this case, the seventh adjunction mapping $\varphi_7$ maps $S_6$ into $\mathbb{P}^4$ and then $S_7$ has degree $14 - r_6$. Furthermore, $9 \leq r_6 \leq r_5 = 10$. If $r_6 = 10$, then $S_7 \subset \mathbb{P}^4$ has degree 4 which means that $H_7 \equiv 3L - \sum_{i=1}^{5} E_i$. Then $H_0 \equiv 24L - \sum_{i=1}^{5} 8E_i - \sum_{j=6}^{10} 7E_j$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. If $r_6 = 9$, then $S_7 \subset \mathbb{P}^4$ has degree 5 which means that $H_7 \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_i$. Then $H_0 \equiv 25l - 9E_1 - \sum_{i=2}^{8} 8E_i - 7E_9 - 6E_{10}$ which has both $\pi_0 = 8$ and $H_0^2 = 11$. □

### 3.4 List of possibilities.

In this section we show a cheap but efficient numerical method to discard divisor classes of a given speciality. Then we apply our method to our results in Proposotion 10 till Proposition 19 and obtain a relatively short list of possible embeddings. First we need a definition.

Let $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_r)$ and let $H$ be a very ample divisor on $S$. We say that a plane curve of degree $d$ and multiplicity $m_i$ at $x_i$ is *$k$-special curve* if $\binom{d+2}{2} \geq (1-k) + \sum \binom{m_i+1}{2}$. Informally, our next result suggests that there should exist $k$-special curves on $S$ whenever $h^1(\mathscr{O}_S(H)) = k$.

**Lemma 20.** *Let $H$ be a very ample divisor on $S$ with a decomposition $H \equiv A + B$. Suppose $h^1(\mathscr{O}_S(H)) + \chi(\mathscr{O}_S(A)) > 0$, $h^2(\mathscr{O}_S(A)) = 0$ and suppose $H.B > 2p_a(B) - 2$. Then $A$ is effective on $S$.*

*Proof.* The result is clear if $\chi(\mathscr{O}_S(A)) > 0$. So suppose $\chi(\mathscr{O}_S(A)) \leq 0$. The assumption $H.B > 2p_a(B) - 2$ implies that $h^1(\mathscr{O}_B(H)) = 0$, by Riemann-Roch. Let $B'$ be any irreducible component of $B$. Taking cohomology of the short exact sequence

$$0 \longrightarrow \mathscr{O}_{B-B'}(H - B') \longrightarrow \mathscr{O}_B(H) \longrightarrow \mathscr{O}_{B'}(H) \longrightarrow 0$$

we get $h^1(\mathscr{O}_{B'}(H)) \leq h^1(\mathscr{O}_B(H)) = 0$. Now we take cohomology of

$$0 \longrightarrow \mathscr{O}_S(A) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_B(H) \longrightarrow 0$$

to get $h^1(\mathscr{O}_S(H)) \leq h^1(\mathscr{O}_S(A))$. Combining the latter with our assumption $h^1(\mathscr{O}_S(H)) + \chi(\mathscr{O}_S(A)) > 0$, we get $h^0(\mathscr{O}_S(A)) > 0$. $\square$

Note that a divisor class $H$ in Proposition 10 till Proposition 19 must have $h^1(\mathscr{O}_S(H)) = 1$ whenever $\varphi_H$ embeds $S \hookrightarrow \mathbb{P}^5$. The idea now is to find 1-special curves on $S$ by using Lemma 20 and then study the numerical invariants of the 1-special curves. For a study of the numerical invariants, recall that every curve $C$ on $S$ has degree $H.C > 0$. Catanese and Franciosci, Proposition 5.2 in [CF93], have improved the lower bound for $H.C$ for curves of small arithmetic genus. We state Catanese and Franciosi's result.

**Proposition 21.** *Suppose $H$ is a very ample divisor on a smooth surface $S$. Then every effective divisor $C$ on $S$ with arithmetic genus $p_a(C) \leq 2$ has degree $H.C \geq 2p_a(C) + 1$. Particulary, if $H.C \leq 3$ then $p_a(C) \leq 1$.*

We are now ready to show the following result.

**Theorem 22.** *Suppose there exists a linearly normal smooth rational surface $S$ of degree 11 and sectional genus 8 embedded in $\mathbb{P}^5$. If $i : S \hookrightarrow \mathbb{P}^5$ is an embedding and $\mathscr{L} \simeq i^*\mathscr{O}_{\mathbb{P}^5}(1)$ is the very ample line bundle associated to $i$, then the associated very ample divisor $H$ of $\mathscr{L}$ is such that $\varphi_{\mathscr{L}\otimes\omega_S^n}(S)$ is a curve for some $n > 0$, $\deg \varphi_{\mathscr{L}\otimes\omega_S^n} = 2$ for some $n > 1$, or $H$ is one of the following divisor classes:*

| $K_S^2$. | *Type.* | $H$. |
|---|---|---|
| $-10.$ | $[6; 2^2, 1^{17}]$. | $6L - \sum_{i=1}^{2} 2E_i - \sum_{j=3}^{19} E_j$. |
| $-8.$ | $[7; 2^7, 1^{10}]$. | $7L - \sum_{i=1}^{7} 2E_i - \sum_{j=3}^{17} E_j$. |
| $-7.$ | $[9; 3^6, 2^2, 1^8]$. | $9L - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{8} 2E_j - \sum_{k=9}^{16} E_k$. |
| $-6.$ | $[10; 4^1, 3^7, 2^1, 1^6]$. | $10L - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{15} E_j$. |

*Proof.* It suffices to show that every divisor class in Proposition 10 til Proposition 19, except the four divisor classes in the statement of the Theorem, can not simultaneously be very ample and have six global sections on $S$. We show this by finding an explicit decomposition $H \equiv A + B$, for each $H$, where $A$ will be a 1-special curve being effective by a use of Lemma 20 and the numerical invariants of $A$ will contradict Proposition 21. We proceed by checking each divisor class in Proposition 10-19 in descending order.

| Type of $H$ | Type of $A$ | $\chi(\mathscr{O}_S(A))$ | $p_a(A)$ | $H.A$ | $\chi(\mathscr{O}_S(B))$ | $p_a(B)$ | $H.B$ |
|---|---|---|---|---|---|---|---|
| $[25; 9^1, 8^7, 7^1, 6^1]_1$ | $[9; 3^8, 2^2]$ | 1 | 2 | 4 | – | – | – |
| $[24; 8^5, 7^5]$ | $[8; 3^5, 2^5]$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[19; 6^9, 5^1, 1^1]$ | $[9; 3^8, 2^2, 1^1]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[16; 5^9, 4^1, 2^1]$ | $[6; 2^9, 1^1]$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[16; 6^1, 5^7, 4^1, 3^2]$ | $[7; 3^1, 2^{10}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[15; 5^5, 4^5, 3^1]$ | $[5; 2^5, 1^6]$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[14; 5^1, 4^{10}]_2$ | $[7; 3^1, 2^{10}]$ | 0 | 2 | 3 | 3 | 4 | 8 |
| $[16; 6^1, 5^7, 4^2, 1^2]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[13; 4^9, 3^1, 2^1, 1^1]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[13; 5^1, 4^7, 3^1, 2^3]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 3 | 3 | 4 | 8 |
| $[12; 4^5, 3^5, 2^2]$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[11; 4^1, 3^{10}, 2^1]_2$ | $[3; 1^{10}]$ | 0 | 1 | 2 | 3 | 5 | 9 |
| $[12; 4^6, 3^3, 2^2, 1^2]_2$ | $[6; 2^8, 1^4]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[11; 4^2, 3^8, 2^1, 1^2]_3$ | $[4; 2^1, 1^{12}]$ | 0 | 2 | 4 | 3 | 3 | 7 |
| $[10; 3^8, 2^4, 1^1]_2$ | $[4; 2^1, 1^{13}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[10; 3^9, 2^1, 1^4]_4$ | $[1; 1^2]$ | 1 | 0 | 4 | 0 | 4 | 7 |
| $[8; 2^{13}, 1^1]_2$ | $[4; 2^1, 1^{12}]$ | 0 | 2 | 4 | 3 | 3 | 7 |
| $[8; 3^1, 2^{10}, 1^4]_2$ | $[4; 2^1, 1^{12}]$ | 0 | 2 | 4 | 2 | 3 | 7 |
| $[8; 3^2, 2^7, 1^7]_{3,4}$ | $[2; 1^5]$ | 1 | 0 | 4 | 0 | 4 | 7 |
| $[9; 3^5, 2^5, 1^5]_2$ | $[3; 1^{10}]$ | 0 | 1 | 2 | 3 | 5 | 9 |

Note that we have written subscripts on some of the types of $H$.
Subscript 1 means that Riemann-Roch yields that $A$ is effective on $S$ but the arithmetic genus and degree contradicts Proposition 21.
Subscript 2 means that one must choose $A$ relative to the ordering $i \geq j$ if and only if $A.E_i \geq A.E_j$. For instance, if $H$ is of type $[9; 3^5, 2^5, 1^5]_2$ then $A \equiv 3L - \sum_{i=1}^{10} E_i$.
Subscript 3 means that we have made a basechange $\operatorname{Pic} \mathbb{F}_0 \to \operatorname{Pic} \mathbb{P}^2$ by embedding $\mathbb{F}_0$ as the quadric surface in $\mathbb{P}^3$ with the Segre embedding.
Subscript 4 occurs in two cases. In type $[10; 3^9, 2^1, 1^4]_4$ we must choose $A \equiv L - E_8 - E_9$. In type $[8; 3^2, 2^7, 1^7]_{3,4}$ we must choose $A \equiv 2L - \sum_{i=1}^{2} E_i - \sum_{j=7}^{9} E_j$. In these cases, $B$ is a 1-special curve and effective by 20 since $H.A > 2p_a(A) - 2$. On the other hand, $H.B > 2p_a(B) - 2$ and $A^2 > 2p_a(A) - 2$ combined contradicts $h^1(\mathscr{O}_S(H)) = 1$.
We illustrate how one may use the table above to obtain a contradiction. If $H$ is of type $[24; 8^5, 7^5]$ and $A$ is of type $[8; 3^5, 2^5]$, then $H \equiv 24L - \sum_{i=1}^{5} 8E_i - \sum_{j=6}^{10} 7E_j$ and $A \equiv 8L - \sum_{i=1}^{5} 3E_i - \sum_{j=6}^{10} 2E_j$. Note that $H.B > 2p_a(B) - 2$. Since $A$ is a 1-special curve, i.e. $\chi(\mathscr{O}_S(A)) = 0$, Lemma 20 implies that $h^0(\mathscr{O}_S(A)) \geq 1$ whenever $h^0(\mathscr{O}_S(H)) = 1$. But then the effectivity of $A$ on $S$ contradicts Proposition 21 since $H.A \leq 2p_a(A)$ and $p_a(A) = 2$. This in turn contradicts the very ampleness of $H$. $\square$

This concludes Chapter 3. In the next two chapters we study the 4 remaining divisor classes in Theorem 22.

# 4 A study of the possibilities.

We state some results on the very ampleness of a divisor class due to Alexander-Bauer and Catanese-Franciosi-Hulek-Reid. Then we use these results to propose a general strategy for constructing smooth rational surfaces of speciality one. After this we show that 2 of the divisor classes in Theorem 22 do not induce embeddings of $S \hookrightarrow \mathbb{P}^5$, under some mild assumptions, We end this chapter by discussing 1 of the divisor classes in Theorem 22.

## 4.1 General strategy.

Given any line bundle $\mathscr{L} = \mathscr{O}_S(H)$ on a smooth rational surface $S \simeq \tilde{\mathbb{P}}^2(x_1, .., x_r)$, it is in general difficult to decide whether $\mathscr{L}$ is very ample on $S$ or not. There is however a versatile result, due to Alexander and Bauer, which provides us with sufficient conditions for $\mathscr{L}$ to be very ample. The idea behind Alexander and Bauer's result is that if $\mathscr{L}$ restricts to a very ample line bundle on a suitable family of curves on $S$ then $\mathscr{L}$ is itself very ample on $S$, given some minor assumptions. This allows us to answer the question of $\mathscr{L}$ being very ample on $S$ by answering the question of $\mathscr{L}$ being very ample on some curves on $S$. We state their precise result, which is Proposition 5.1 in [CF93] and Lemma 0.12 in [Ran88].

**Lemma 23 (Alexander-Bauer).** *An effective line bundle $\mathscr{O}_S(H) \simeq \mathscr{O}_S(A_1 + A_2)$ is very ample on $S$, if each one of the following holds:*

| | | |
|---|---|---|
| *(1).* | $h^0(\mathscr{O}_S(A_i)) \geq 2$, | *for some* $1 \leq i \leq 2$. |
| *(2).* | $\mathscr{O}_A(H)$ *is very ample,* | *for all* $A \in \lvert A_1 \rvert \cup \lvert A_2 \rvert$. |
| *(3).* | $H^0(\mathscr{O}_S(H)) \to H^0(\mathscr{O}_{A_i}(H))$ *is surjective,* | *for all* $1 \leq i \leq 2$. |

The reader may have noticed that the Alexander-Bauer lemma does not give any information on how to determine if $\mathscr{L}$ restricts to a very ample line bundle on curves. Recall that it follows from Riemann-Roch that $\mathscr{L}$ is very ample on an irreducible curve $C$ if $\deg(\mathscr{L} \otimes \mathscr{O}_C) \geq 2p_a(C) + 1$. Catanese, Franciosi, Hulek and Reid, Theorem 1.1 in [CFHR99], have generalized the latter into a result which also includes reducible curves. The part of their result which we shall be using is:

**Theorem 24 (Curve embedding).** *Let $A$ and $H$ be effective divisors on a surface $S$. Then $\mathscr{O}_A(H)$ is very ample whenever $H.A' \geq 2p_a(A') + 1$, for all subcurves $A' \subset A$.*

Combining the two results above it is fairly straightforward to determine whether $\mathscr{L} \simeq \mathscr{O}_S(H)$ is very ample on $S$ or not. This can be done by finding an effective decomposition $H \equiv A_1 + A_2$ which satisfies the assumptions of Lemma 23.1 and Lemma 23.3. Then we may assume that no subcurve $A_i' \subset A_i$ satisfies $H.A_i' \leq 2p_a(A_i')$, for both $1 \leq i \leq 2$. The latter would then give us open conditions on the choice of the points $x_1, .., x_r \in \mathbb{P}^2$. This would then automatically secure that $\mathscr{L}$ is very ample on $S$ by Lemma 23 and Theorem 24. In fact, if the decomposition $H \equiv A_1 + A_2$ is chosen such that every subcurve $A_i' \subset A_i$ has arithmetic genus $p_a(A_i') \leq 2$, then Proposition 21 secures that the choice of points is the unique configuration of the points $x_1, .., x_r$ which yield that $\mathscr{L}$ is very ample on $S$. This is the strategy Catanese and Franciosi use in [CF93], where they consider non-special surfaces of degree $\leq 8$ in $\mathbb{P}^4$ or the strategy used by Catanese and Hulek in [CH97], where they consider the non-special surface of degree 9 in $\mathbb{P}^4$.

There is however, in general, a drawback to the strategy mentioned above. The approach above yields an embedding $\varphi_H : S \hookrightarrow \mathbb{P}^n$, where $n = \chi(\mathscr{O}_S(H)) - 1$, such that the one is indirectly excluding the cases of special surfaces. In the case of special surfaces, it is a nontrivial task to construct surfaces by the method above as we shall see later in this section and in Chapter 5. We will therefore discuss how to generalize the strategy above to surfaces of speciality one, i.e. when $h^1(\mathscr{O}_S(H)) = 1$. We will see that we neccessarily must have closed conditions for the choice of the points $x_1, .., x_r \in \mathbb{P}^2$. Our reason for considering $h^0(\mathscr{O}_S(H)) = 1$ is that this is exactly the speciality we must have in Theorem 22.

We begin by observing the following.

**Lemma 25.** *Let $H \equiv A + B$ be an effective divisor on a smooth surface $S$ such that $h^1(\mathscr{O}_S(A)) = 0$ and $H.B = 2p_a(B) - 2$. Then $h^1(\mathscr{O}_S(H)) = 1$ if and only if $\mathscr{O}_B(H) \simeq \omega_B$.*

*Proof.* The idea is to compare the dimensions of the cohomology groups associated to the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(A) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_B(H) \longrightarrow 0.$$

Note that $h^1(\mathscr{O}_S(H)) = h^1(\mathscr{O}_B(H))$ since $h^i(\mathscr{O}_S(A)) = 0$ for $i > 0$. If $\mathscr{O}_B(H) \simeq \omega_B$, then $h^1(\mathscr{O}_S(H)) = h^1(\omega_B) = 1$. Conversely, if $h^1(\mathscr{O}_S(H)) = 1$, then $h^1(\mathscr{O}_S(H)) = h^0(\omega_B \otimes \mathscr{O}_B(-H)) = 1$ by Serre duality such that $(K_B - H)_{|B}$ is effective. Taking cohomology of

$$0 \longrightarrow \mathscr{O}_B \longrightarrow \omega_B \otimes \mathscr{O}_B(-H) \longrightarrow \mathscr{F} \longrightarrow 0,$$

where $\mathscr{F}$ is a zero-dimensional scheme supported on $B \cap (K_B - H)$, we see that $h^0(\mathscr{F}) = 0$ by taking Euler characteristics of the sequence above. Then $\mathrm{Supp}(\mathscr{F}) = \emptyset$ such that $\mathscr{F}$ is the zero sheaf. Then the vanishing of the stalks $\mathscr{F}_p$, for all $p \in B$, yields that $\mathscr{O}_B \simeq \omega_B \otimes \mathscr{O}_B(-H)$ or equivalently $\omega_B \simeq \mathscr{O}_B(H)$. $\square$

Due to the usefulness of a decomposition as described in Lemma 25 we make the following definition.

**Definition 26.** Let $H$ be a effective divisor on $S$. We say that $H \equiv A_1 + A_2$ is a *nice decomposition* of $H$ on $S$ if $A_1$ and $A_2$ are both effective divisors on $S$ such that $A_1$ is non-special on $S$ and $H.A_2 = 2p_a(A_2) - 2$.

Note that there is a priori no reason to assume that there exists a nice decomposition for a given divisor $H$. It is however, in our cases, a computational matter to verify the existence directly by finding explicit nice decompositions. Before we continue any further we need two lemmas which will aid us in the search for nice decompositions. To secure non-speciality of one of the components of $H$ we use the following.

**Lemma 27.** *Let $A$ be an effective divisor on a smooth rational surface $S$. Suppose $A^2 > 2p_a(A) - 2$. Then $h^1(S, \mathscr{O}_S(A)) = 0$.*

*Proof.* Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_S \longrightarrow \mathscr{O}_S(A) \longrightarrow \mathscr{O}_A(A) \longrightarrow 0.$$

Upon taking cohomology we get $H^i(\mathscr{O}_S(A)) \simeq H^i(\mathscr{O}_A(A))$, for each $i > 0$. Therefore $h^1(\mathscr{O}_S(A)) = h^1(\mathscr{O}_A(A))$. Now if $A^2 > 2p_a(A) - 2$, then $h^1(\mathscr{O}_A(A)) = 0$ by Riemann-Roch and the lemma follows. □

To ensure that $H$ restricts to the canonical divisor on one of the components we will be using the following.

**Lemma 28.** *Let $H \equiv A_1 + A_2$ be a nice decomposition of $H$ on a surface $S$. Suppose $A_2$ is smooth, the intersection product $H.A_2 = 2p_a(A_2) - 2$ and the divisor $(A_1 - K_S)_{|A_2}$ is effective on $A_2$. Then $\mathscr{O}_{A_2}(H) \simeq \omega_{A_2}$.*

*Proof.* Since $A_2$ is smooth and effective on $S$, the adjunction formula applies and tells us that $\omega_{A_2} \simeq \mathscr{O}_{A_2}(A_2 + K_S)$. Note that it suffices to show that $\mathscr{O}_{A_2}(A_1 - K_S) \simeq \mathscr{O}_{A_2}$. Since $\mathscr{O}_{A_2}(A_1 - K_S)$ is effective on $A_2$, we have

$$0 \longrightarrow \mathscr{O}_{A_2} \longrightarrow \mathscr{O}_{A_2}(A_1 - K_S) \longrightarrow \mathscr{F} \longrightarrow 0,$$

where $\mathscr{F}$ is supported on $A_2 \cap (A_1 - K_S)$. Combining $H.A_2 = 2p_a(A_2) - 2$ with the adjunction formula we get $A_2.(A_1 - K_S) = 0$. Therefore $h^0(\mathscr{F}) = 0$ such that $\mathscr{O}_{A_2} \simeq \mathscr{O}_{A_2}(A_1 - K_S)$ or equivalently $\mathscr{O}_{A_2}(H) \simeq \mathscr{O}_{A_2}(A_2 + K_S)$. Now the lemma follows from the adjunction formula. □

We are now ready to sketch our general strategy for finding explicit open and closed conditions imposed upon the points $x_1, .., x_r \in \mathbb{P}^2$. The basic idea in our strategy can be divided into two parts. First part, we describe how we may use Lemma 23 till Lemma 25 to find sufficient conditions for the very ampleness and speciality 1 of a given divisor class. The sufficient conditions will be statements about curves and complete linear systems on $S$. Second part, we propose a method for translating statements in the first part into statements about the positioning of the points $x_1, .., x_r$.

For the first part of our general strategy we observe the following.

**Observation I.** *Given a nice decomposition $H \equiv A_1 + A_2$ of an effective divisor on $S$ with $H.A_2 = 2p_a(A_2) - 2$. Then there at least* 1 *and at most* 3 *closed conditions implying that $H$ is very ample and $h^0(\mathscr{O}_S(H)) = 1$. Explicitly, the open and closed conditions form a subset of the following conditions, where $1 \le i \le 2$ and $k \in \mathbb{Z}_{\ge 0}$:*

*(C1).* $\mathscr{O}_{A_2}(H) \simeq \omega_{A_2}$.

*(C2)$_i$.* $\dim |A_i| := \chi(\mathscr{O}_S(H)) - h^0(\mathscr{O}_{H-A_i}(H))$.

*(O1)$_k$.* *No proper subcurve $A'_k$ of a curve in $|A_1| \cup |A_2|$ satisfies $H.A'_k \le 2p_a(A'_k)$.*

Suppose an effective divisor $H$ on $S$ is given and that $H \equiv A_1 + A_2$ defines a nice decomposition of $H$. The first natural closed condition is then imposed upon by Lemma 25, forcing us to assume $H_{|A_2} \equiv K_{A_2}$ to obtain a sufficient condition for $h^1(\mathscr{O}_S(H))$. This is the closed condition C1 in Observation I. To be able to use the Alexander-Bauer lemma, we must then set $\dim |A_i|$ to be as described in the closed condition C$2_i$ in Observation I, for $1 \leq i \leq 2$. Otherwise, at least one of the curves $A_i$ would not be effective on $S$ or at least one of the restriction maps $H^0(\mathscr{O}_S(H)) \to H^0(\mathscr{O}_{A_i}(H))$ would not be surjective. Note that if $\chi(\mathscr{O}_S(H)) < h^0(\mathscr{O}_{A_i}(H))$, for some $i$, then the closed condition C$2_i$ can never be satisfied. If this is the case, then one may replace the decomposition $H \equiv A_1 + A_2$ with another decomposition on which it is possible to assume C$2_i$. So we may suppose it is possible to choose a nice decomposition $H \equiv A_1 + A_2$ satisfying the three closed conditions C1, C$2_1$ and C$2_2$, and thus showing the statement about closed conditions in Observation I. Then the only assumption we need to take care of in the Alexander-Bauer lemma is the assumption about very ampleness.

Suppose a nice decomposition $H \equiv A_1 + A_2$ on $S$ satisfies all assumptions in the Alexander-Bauer lemma except possibly the assumption in Lemma 23.2 and suppose $H_{|A_2} \equiv K_{A_2}$. For the very ampleness, we note that if $p_a(A_1) \leq 2$, then we may assume the open conditions O$1_k$ are true for suitably chosen proper subcurves $A'_k \subset A_1$. Then Theorem 24 implies that $|H|$ embeds every member of $|A_1|$. So, we may assume $p_a(A_1) \leq 2$ by possibly replacing the nice decomposition $H \equiv A_1 + A_2$ with another nice decomposition. To show that $|K_{A_2}|$ is very ample one could possibly show that $A_2$ is non-hyperelliptic but we sketch a different idea. Suppose $A_2$ is smooth. Then the adjunction formula applies and yields $H_{|A_2} \equiv (A_2 + K_S)|_{|A_2}$. Taking cohomology of

$$0 \longrightarrow \mathscr{O}_S(K_S) \longrightarrow \mathscr{O}_S(A_2 + K_S) \longrightarrow \mathscr{O}_{A_2}(H) \longrightarrow 0,$$

it follows that the restriction map $H^0(\mathscr{O}_S(A_2 + K_S)) \to H^0(\mathscr{O}_{A_2}(H))$ is surjective, since the rationality of $S$ implies that $h^i(\mathscr{O}_S(K_S)) = 0$ for $0 \leq i \leq 1$. Then one may possibly use the Alexander-Bauer lemma on $\mathscr{O}_S(A_2 + K_S)$ whenever it is possible.

This concludes the first part of our main idea for obtaining and verifying that our open and closed conditions in Observation I yield $h^0(\mathscr{O}_S(H)) = 1$ and the very ampleness of $H$.

Note that the method described above applies to any choice for $H^2$ and $\pi_S$, as long as one is requiring $h^0(\mathscr{O}_S(H)) = 1$. Also note that it is possible to avoid the closed conditions C$2_i$ by possibly changing to a different nice decomposition for $H$. In fact, the closed condition C$1_2$ is redundant since $h^0(\mathscr{O}_S(A_1)) = 0$. On the other hand, note that the closed condition C1 is unavoideable due to Lemma 25. Finally, note that the open conditions O$1_k$ are direct statements about the positioning of the points $x_1, .., x_r \in \mathbb{P}^2$.

Next we consider the second part of our general strategy, i.e. we discuss how to find sufficient conditions implying that the conditions C1 and C$2_i$, mentioned in Observation I, are true. This idea stems from Ranestad's [Ran88] construction of a smooth rational surface of degree 10 and sectional genus 8 in $\mathbb{P}^4$.

**Observation II.** *Given a nice decomposition $H \equiv A_1 + A_2$ of an effective divisor on $S$ with $H.A_2 = 2p_a(A_2) - 2$. Let $\pi : S \to \mathbb{P}^2$ denote the morphism obtained by blowing up $x_1, .., x_n, y_1, .., y_m \in \mathbb{P}^2$ and denote $E_i := \pi^{-1}(x_i)$ and $F_i := \pi^{-1}(y_j)$. Then any effective divisor $D$ on $S$ such that $\pi(A_2) \cap \pi(D) = \sum x_i + \sum y_i + \sum y_i'$, where $y_i'$ are common tangent directions of $\pi(A_2)$ and $\pi(D)$ at $y_i$, implies that $\mathscr{O}_{A_2}(D) \simeq \mathscr{O}_{A_2}(\sum F_i)$. In particular, if $A_2$ is smooth and $D \equiv A_1 - K_S + \sum F_i$ then any such configuration of $x_1, .., x_n, y_1, .., y_m$ implies that the closed condition C1 holds, i.e. $\mathscr{O}_{A_2}(H) \simeq \omega_{A_2}$.*

The first conclusion in Observation II comes from the fact that by blowing up the points $x_1, .., x_n, y_1, .., y_m$ to obtain $S$, the tangent directions $\sum y_i'$ correspond to points on the exceptional divisors $\sum F_i$ such that $D|_{A_2} \equiv \sum F_i$. The second conclusion follows from the adjunction formula, since then we would have $\mathscr{O}_{A_2}(A_1 - K_S + \sum F_i) \simeq \mathscr{O}_{A_2}(\sum F_i)$, or equivalently $\mathscr{O}_{A_2}(H) \simeq \mathscr{O}_{A_2}(A_2 + K_S)$.

The idea now is this. Given a nice decomposition $H \equiv A_1 + A_2$ one may determine $b_1, .., b_m \in \{0, 1\}$ such that $A_2.(A_1 - K_S + \sum b_i F_i) = \sum b_i$, for this would then yield a part of a closed condition which one implies that the closed condition C1 is true. After determining possibilities for $b_1, .., b_m$, one may rule out the curves $A_1 - K_S + \sum b_i F_i$ contradicting some open condition $\text{O1}_k$, for some $k$. Note that one also has to make sure that the curves $A_1 - K_S + \sum b_i F_i$ and their residual curves in $|H|$ do not contradict the closed conditions $\text{C2}_i$. If there are no possibilities, then one may switch to another nice decomposition for $H$ and repeat the procedure above with the new nice decomposition.

Now, assume that there exists a nice decomposition $H \equiv A_1 + A_2$ which admits a replacement of the closed condition C1 as described in Observation II. To translate the closed conditions $\text{C2}_i$ into statements about $x_1, .., x_n, y_1, .., y_m$, recall that the closed condition $\text{C2}_2$ is automatically satisfied, for all nice decompositions. So it suffices to consider the condition $\text{C2}_1$. To translate $\text{C2}_1$, we try to find nice decompositions such that C1 implies that $\text{C2}_1$ holds and thus reducing ourselves to considering C1 instead. An example of this may be found in Lemma 45 in the proof of the main theorem.

This finishes our strategy for translating the open and closed conditions in Observation I into statements about points in the projective plane.

Before we conclude this section we make two remarks. First, note that our general strategy is not very fruitful if there are few or none nice decompositions. Second, note that our strategy may not apply directly when $h^0(\mathscr{O}_S(H)) > 1$ due to Lemma 25.

## 4.2 Indications of non-existence.

In this section we show that 2 of the complete linear systems in Theorem 22, namely the linear systems of type $[6; 2^2, 1^{17}]$ and $[10; 4^1, 3^7, 2^1, 1^6]$, do not induce an embedding of $S$ into $\mathbb{P}^5$, under some mild assumptions. First we consider $[6; 2^2, 1^{17}]$.

**Proposition 29.** *Let $S$ be a rational surface with $K_S^2 = -10$. Let $\pi : S \longrightarrow \mathbb{P}^2$ denote the morphism obtained by blowing up the points $x_1, x_2, y_1, .., y_{17} \in \mathbb{P}^2$ and let $E_i := \pi^{-1}(x_i)$, $F_i = \pi^{-1}(y_i)$ and $L := \pi^* l$ where $l \subset \mathbb{P}^2$ is a line. Suppose that the divisor class*

$$H \equiv 6L - \sum_{i=1}^{2} 2E_i - \sum_{j=1}^{17} F_j$$

*has $h^0(\mathscr{O}_S(H)) = 6$ and suppose that the curves*

$$A_{rs} \equiv 5L - \sum_{i=1}^{2} 2E_i - \sum_{j=1}^{17} F_j + E_r + F_s$$

*are smooth on $S$, for all $1 \leq r \leq 2$, $1 \leq s \leq 17$. Then $H$ is not very ample on $S$.*

*Proof.* We begin by noticing that the following open conditions are neccessary for the very ampleness of $H$:

| | | |
|---|---|---|
| (O1). | $\lvert L - \sum_{i \in I} E_i - \sum_{j \in J} F_j \rvert = \emptyset,$ | where $2\lvert I\rvert + \lvert J\rvert \geq 6.$ |
| (O2). | $\lvert 2L - \sum_{i \in I} E_i - \sum_{j \in J} F_j \rvert = \emptyset,$ | where $2\lvert I\rvert + \lvert J\rvert \geq 12.$ |
| (O3). | $\lvert 4L - \sum_{i=1}^{2} E_i - \sum_{j=1}^{17} F_j \rvert = \emptyset.$ | |

The conditions O1-O2 are neccessary since any curve $C$ as in the complete linear systems of O1 or O2 intersects non-positive with $H$. Note that if O3 was non-empty, then there would exist a effective curve $C$ on $S$ such that $H.C = 3$ and $p_a(C) = 3$. The latter would contradict the very ampleness of $H$ since $H.C = 3$ implies that $p_a(C) \leq 1$, by Proposition 21.

We proceed with the proof. Using Riemann-Roch, the curves $A_{rs}$ are all effective on $S$ since $\chi(\mathscr{O}_S(A_{rs})) = 1$. Denote the divisor class of the residual curve of $A_{rs}$ in $|H|$ by

$$B_{rs} \equiv H - A_{rs} \equiv L - E_r - F_s.$$

**Lemma 30.** *$\mathscr{O}_{A_{rs}}(B_{rs} - K_S) \simeq \mathscr{O}_{A_{rs}}$, for all $r, s$.*

*Proof.* Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(B_{rs}) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_{A_{rs}}(H) \longrightarrow 0.$$

Taking cohomology and noting that $B_{rs}^2 > 2p_a(B_{rs}) - 2$ we get $h^1(\mathscr{O}_S(B_{rs})) = 0$. Therefore, $h^1(\mathscr{O}_S(H)) = h^1(\mathscr{O}_{A_{rs}}(H)) = 1$ since $h^0(\mathscr{O}_S(H)) = 6$. Using Serre duality we have $h^1(\mathscr{O}_{A_{rs}}(H)) = h^0(\mathscr{O}_{A_{rs}}(K_{A_{rs}} - H)) = 1$. Since $A_{rs}$ is smooth, the adjunction formula tells us that $K_{A_{rs}} \equiv A_{rs} + K_S$ such that $h^0(\mathscr{O}_{A_{rs}}(K_S - B_{rs})) = 1$. Now, Lemma 28 implies that $\mathscr{O}_{A_{rs}}(H) \simeq \omega_{A_{rs}}$ since $H.A_{rs} = 2p_a(A_{rs}) - 2$. Another use of the adjunction formula yields $\mathscr{O}_{A_{rs}}(H) \simeq \mathscr{O}_{A_{rs}}(A_{rs} + K_S)$. Twisting the former with $\mathscr{O}_{A_{rs}}(-A_{rs} - K_S)$ we obtain $\mathscr{O}_{A_{rs}}(B_{rs} - K_S) \simeq \mathscr{O}_{A_{rs}}$. $\square$

**Lemma 31.** *$h^0(\mathscr{O}_S(L - K_S + F_s)) > 0$, for all $s$.*

*Proof.* Without loss of generality we may assume that $r = 2$ such that $A_s := A_{2,s}$ and $B_s := B_{2,s}$. Then we have a short exact sequence

$$0 \longrightarrow \mathscr{O}_S(B_s - K_S - A_s) \longrightarrow \mathscr{O}_S(B_s - K_S) \longrightarrow \mathscr{O}_{A_s} \longrightarrow 0$$

due to Lemma 30. Twisting the sequence above with $\mathscr{O}_S(E_2 + 2F_s)$ we get

$$0 \longrightarrow \mathscr{O}_S(-L + E_1) \longrightarrow \mathscr{O}_S(L - K_S + F_s) \longrightarrow \mathscr{O}_{A_s}(E_2 + 2F_s) \longrightarrow 0.$$

Note that $h^0(\mathscr{O}_S(-L + E_1)) = 0$. Moreover, we claim that $h^1(\mathscr{O}_S(-L + E_1)) = 0$. The latter can be seen by taking cohomology of

$$0 \longrightarrow \mathscr{O}_S(-L + E_1) \longrightarrow \mathscr{O}_S \longrightarrow \mathscr{O}_{L-E_1} \longrightarrow 0.$$

Since $S$ is a rational surface we have $h^0(\mathscr{O}_S) = 1$ and $h^1(\mathscr{O}_S) = 0$. Combining $\chi(\mathscr{O}_{L-E_1}) = 1$ and $h^1(\mathscr{O}_{L-E_1}) = g_{L-E_1} = 0$ we obtain $h^0(\mathscr{O}_{L-E_1}) = 1$. Then we have $h^0(\mathscr{O}_{L-E_1}) - h^0(\mathscr{O}_S) = h^1(\mathscr{O}_S(-L + E_1)) = 0$. In particular, this means that

$$h^0(\mathscr{O}_S(L - K_S + F_s)) = h^0(\mathscr{O}_{A_s}(E_2) \otimes \mathscr{O}_{A_s}(F_s) \otimes \mathscr{O}_{A_s}(F_s))$$

by the second short exact sequence above. Next we claim that $\mathscr{O}_{A_s}(F_s) \simeq \mathscr{O}_{A_s}$. To see this, consider

$$0 \longrightarrow \mathscr{O}_{A_s} \longrightarrow \mathscr{O}_{A_s}(F_s) \longrightarrow \mathscr{F} \longrightarrow 0$$

where $\mathscr{F}$ is a zero-dimensional scheme supported on the scheme-theoretic intersection $A_s \cap F_s$. Since $A_s.F_s = 0$, we get $h^0(\mathscr{F}) = 0$, by taking Euler characteristics of the sequence above, such that $\mathscr{O}_{A_s} \simeq \mathscr{O}_{A_s}(F_s)$. This yields that

$$h^0(\mathscr{O}_S(L - K_S + F_s)) = h^0(\mathscr{O}_{A_s}(E_2)).$$

Now we take cohomology of the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(E_2 - A_s) \longrightarrow \mathscr{O}_S(E_2) \longrightarrow \mathscr{O}_{A_s}(E_2) \longrightarrow 0$$

and notice that $h^0(\mathscr{O}_S(E_2)) \leq h^0(\mathscr{O}_{A_s}(E_2))$ since $h^0(\mathscr{O}_S(E_2 - A_s)) = 0$. Clearly $E_2$ is effective on $S$ such that we must have $h^0(\mathscr{O}_S(L - K_S + F_s)) > 0$. ☐

We are ready to show the Proposition. Denote

$$C_s := L - K_S + F_s \equiv 4L - \sum_{i=1}^{2} E_i - \sum_{j=1}^{17} F_j + F_s$$

where $1 \leq s \leq 17$. The images $\pi(C_s)$ of the curve $C_s$ under the blow-down morphism $\pi$ are plane quartics passing through $x_1, x_2$ and 16 of the points $y_1, .., y_{17}$. Then Bezout's theorem implies that the curves $\pi(C_s)$ have a fixed component, since any two $\pi(C_s)$ and

$\pi(C_{s'})$ pass through $x_1, x_2$ and 15 of the points $y_1, .., y_{17}$. This means that the points $x_1, x_2, y_1, .., y_{17}$ all lie on a plane quartic curve such that the curve

$$C \equiv 4L - \sum_{i=1}^{2} E_i - \sum_{j=1}^{17} F_j$$

is effective on $S$. Suppose $H$ is very ample. Note that $H.C = 3$ such that $|H|$ embeds $C$ a cubic curve on $S$. If $C$ is irreducible then we obtain a contradiction on the very ampleness of $H$, because of O3. So, suppose $C$ is reducible on $S$. Then there are two possibilities:

(1). $C$ is the union of three lines $L_1, L_2, L_3$.

(2). $C$ is the union of a conic $Q$ and a line $L'$.

In case (1), by the pigeonhole principle, at least one of the lines $L_i$ passes through at least $\lceil \frac{19}{3} \rceil = 7$ of the exceptional divisors. Then O1 contradicts the very ampleness of $H$. In case (2), again by the pigeonhole principle, either $Q$ passes through at least 12 of the exceptional divisors or $L'$ passes through at least 8 of the exceptional divisors, such that O2 and O1 contradict the very ampleness of $H$. This proves the Proposition. $\square$

Second, we consider the divisor class of type $[10; 4^1, 3^7, 2^1, 1^6]$.

**Proposition 32.** *Let $S$ be a rational surface with $K_S^2 = -6$. Let $\pi : S \longrightarrow \mathbb{P}^2$ denote the morphism obtained by blowing up the points $x_1, .., x_{15} \in \mathbb{P}^2$ and let $E_i := \pi^{-1}(x_i)$ and $L := \pi^* l$ where $l \subset \mathbb{P}^2$ is a line. Suppose that the divisor class*

$$H \equiv 10L - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{15} E_j$$

*has $h^0(\mathscr{O}_S(H)) = 6$ and let*

$$A \equiv 7L - 3E_1 - \sum_{i=2}^{9} 2E_i - \sum_{j=10}^{15} E_j.$$

*Then $A$ is effective on $S$. Furthermore, if $A$ is smooth then $H$ is not very ample on $S$.*

*Proof.* We begin by noticing that the following open condition is neccessary for the very ampleness of $H$ :

(O1). $\quad |6L - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{15} E_j| = \emptyset.$

If there were a curve $C$ in the complete linear system depicted in O1, the curve $C$ would have $p_a(C) = 2$ and $H.C = 2$ which contradicts Proposition 21. Note that $\chi(\mathscr{O}_S(A)) = 0$ such that Riemann-Roch does not justify the effectiveness of $A$ on $S$. However, the next lemma shows that $A$ is indeed effective on $S$. Denote

$$B \equiv H - A \equiv 3L - \sum_{i=1}^{8} E_i$$

as the divisor class of the residual curve of $A$ in $|H|$.

**Lemma 33.** $h^0(\mathscr{O}_S(A)) > 0$.

*Proof.* We take cohomology of the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(A) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_B(H) \longrightarrow 0$$

and note that $H.B > 2p_a(B) - 2$ such that $h^1(\mathscr{O}_B(H)) = 0$. The assumption that $h^0(\mathscr{O}_S(H)) = 6$ implies that $h^1(\mathscr{O}_S(H)) = 1$. Then the surjectivity of $H^1(\mathscr{O}_S(A)) \longrightarrow H^1(\mathscr{O}_S(H))$ implies that $h^1(\mathscr{O}_S(A)) = h^0(\mathscr{O}_S(A)) \geq 1$. ☐

**Lemma 34.** $\mathscr{O}_A(B - K_S) \simeq \mathscr{O}_A$.

*Proof.* Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(B) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_A(H) \longrightarrow 0.$$

Taking cohomology and noting that $B^2 > 2p_a(B) - 2$ we get $h^1(\mathscr{O}_S(B)) = 0$. Therefore $h^1(\mathscr{O}_S(H)) = h^1(\mathscr{O}_A(H)) = 1$ since $h^0(\mathscr{O}_S(H)) = 6$. Using Serre duality we have $h^1(\mathscr{O}_A(H)) = h^0(\mathscr{O}_A(K_A - H)) = 1$. Since $A$ is smooth, the adjunction formula tells us that $K_A \equiv A + K_S$ such that $h^0(\mathscr{O}_A(K_S - B)) = 1$. Now, Lemma 28 implies that $\mathscr{O}_A(H) \simeq \omega_A$ since $H.A = 2p_a(A) - 2$. Another use of the adjunction formula yields $\mathscr{O}_A(H) \simeq \mathscr{O}_A(A + K_S)$. Twisting the former with $\mathscr{O}_A(-A - K_S)$ we obtain $\mathscr{O}_A(B - K_S) \simeq \mathscr{O}_A$. ☐

**Lemma 35.** $h^0(\mathscr{O}_S(B - K_S)) > 0$.

*Proof.* Using Lemma 34 we have an short exact sequence

$$0 \longrightarrow \mathscr{O}_S(B - A - K_S) \longrightarrow \mathscr{O}_S(B - K_S) \longrightarrow \mathscr{O}_A \longrightarrow 0.$$

Note that $h^0(\mathscr{O}_S(B - A - K_S)) = 0$. We claim that $B - A - K_S \equiv -L + E_1 + E_9$ is non-special on $S$, i.e. $h^1(\mathscr{O}_S(-L + E_1 + E_9)) = 0$. To see this, look at the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(-L + E_1 + E_9) \longrightarrow \mathscr{O}_S \longrightarrow \mathscr{O}_{L-E_1-E_9} \longrightarrow 0.$$

Since $h^0(\mathscr{O}_S) = 1$ and $h^1(\mathscr{O}_S) = 0$, due to the rationality of $S$, the only possibility for $h^1(\mathscr{O}_S(-L + E_1 + E_9)) > 0$ is if and only if $h^0(\mathscr{O}_{L-E_1-E_9}) > 1$. But the latter is false since $h^0(\mathscr{O}_{L-E_1-E_9}) = 1$. So $h^1(\mathscr{O}_S(-L + E_1 + E_9)) = 0$. Then the first short exact sequence implies that $h^0(\mathscr{O}_S(B - K_S)) = h^0(\mathscr{O}_A) = 1$. ☐

Lemma 35 implies that the complete linear system in O1 is non-empty. But this contradicts the very ampleness of $H$. This proves the Proposition. ☐

### 4.3 Indications of existence.

In this section we study the divisor class of type $[9; 3^2, 2^2, 1^8]$ in Theorem 22.

**Proposition 36.** *Let $S$ be a rational surface with $K_S^2 = -7$. Let $\pi : S \longrightarrow \mathbb{P}^2$ denote the morphism obtained by blowing up the points $x_1, .., x_6, y_1, y_2, z_1, .., z_8 \in \mathbb{P}^2$ and let $E_i := \pi^{-1}(x_i)$, $F_i = \pi^{-1}(y_i)$, $G_i := \pi^{-1}(z_i)$ and $L := \pi^* l$ where $l \subset \mathbb{P}^2$ is a line. Suppose that the divisor class*

$$H \equiv 9L - \sum_{i=1}^{6} 3E_i - \sum_{j=1}^{2} 2F_j - \sum_{k=1}^{8} G_k$$

*is very ample and suppose $h^0(\mathscr{O}_S(H)) = 6$. Then the divisors*

$$A \equiv 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=1}^{2} F_j - \sum_{k=1}^{8} G_k$$

$$A_1 \equiv 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=1}^{2} F_j - \sum_{k=1}^{8} G_k - F_1$$

$$A_2 \equiv 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=1}^{2} F_j - \sum_{k=1}^{8} G_k - F_2$$

*are all effective on $S$. Moreover, if $A$ is smooth then the points $x_1, .., x_6, y_1, y_2, z_1, .., z_8$ lie on the set-theoretic complete intersection of the curves $A_1$ and $A_2$, that is*

$$A_1 \cap A_2 = \sum x_i + \sum y_i + \sum z_i.$$

*Proof.* Denote the divisor class of the residual curve of $A$ in $|H|$ by

$$B \equiv 3L - \sum_{i=1}^{6} E_i - \sum_{j=1}^{2} F_j.$$

We begin by noting that $\chi(\mathscr{O}_S(A)) = 0$ such that Riemann-Roch does not imply that $A$ is effective on $S$. However, since $A$ is a 1-special curve and $H.B > 2p_a(B) - 2$ it follows from Lemma 20 that $A$ is indeed effective on $S$, i.e. $h^0(\mathscr{O}_S(A)) > 0$.

**Lemma 37.** *$\mathscr{O}_A(A) \simeq \mathscr{O}_A(F_1 + F_2)$.*

*Proof.* Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(B) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_A(H) \longrightarrow 0.$$

Upon taking cohomology we note that $B^2 > 2p_a(B) - 2$ such that $h^1(\mathscr{O}_S(B)) = 0$. Then $h^1(\mathscr{O}_S(H)) = h^1(\mathscr{O}_A(H)) = h^0(\mathscr{O}_A(K_A - H)) = 1$ by our assumption that $h^0(\mathscr{O}_S(H)) = 6$ and by Serre duality. Since $H.A = 2p_a(A) - 2$, Lemma 28 implies that $\mathscr{O}_A(H) \simeq \omega_A$. The smoothness of $A$ yields that $\mathscr{O}_A(H) \simeq \mathscr{O}_A(A + K_S)$ or equivalently $\mathscr{O}_A(A) \simeq \mathscr{O}_A(A - B + K_S)$. Then the lemma follows by noting that $(A - B + K_S)|_A \equiv (F_1 + F_2)|_A$. $\square$

**Lemma 38.** *$h^0(\mathscr{O}_S(A - F_i)) > 0$, for both $1 \le i \le 2$.*

*Proof.* We take cohomology of the sequence

$$0 \longrightarrow \mathscr{O}_S(-F_i) \longrightarrow \mathscr{O}_S(A - F_i) \longrightarrow \mathscr{O}_A(A - F_i) \longrightarrow 0.$$

The Lemma is clear if $h^0(\mathscr{O}_S(-F_i)) > 0$. So suppose $h^0(\mathscr{O}_S(-F_i)) = 0$. Note that $h^2(\mathscr{O}_S(-F_i)) = 0$ by Serre duality. Combining the latter with $\chi(\mathscr{O}_S(-F_i)) = 0$, we obtain $h^1(\mathscr{O}_S(-F_i)) = 0$. Then the sequence above yields that

$$h^0(\mathscr{O}_S(A - F_i)) = h^0(\mathscr{O}_A(A - F_i)).$$

Using Lemma 37 we see that $\mathscr{O}_A(A - F_i) \simeq \mathscr{O}_A(F_j)$ where $j \neq i$ and $1 \le j \le 2$. Then

$$0 \longrightarrow \mathscr{O}_S(F_j - A) \longrightarrow \mathscr{O}_S(F_j) \longrightarrow \mathscr{O}_A(F_j) \longrightarrow 0$$

implies that $h^0(\mathscr{O}_S(F_j)) \le h^0(\mathscr{O}_A(F_j))$ since $h^0(\mathscr{O}_S(F_j - A)) = 0$. It is clear that $F_j$ is effective on $S$ such that we must have $h^0(\mathscr{O}_S(A - F_i)) > 0$. □

The statement about the set-theoretic complete intersection holds since $A_1.A_2 = 0$. This proves the Proposition. □

We comment on our work on this divisor class. In the proof of the Proposition above we met upon three nice decompositions of $H$. Namely,

$$H \equiv A + B \text{ and } H \equiv A_i + (B + F_i), \text{ where } 1 \le i \le 2.$$

Note that the decompositions $H \equiv A_i + (B + F_i)$ satisfy $h^0(\mathscr{O}_{B+F_i}(H)) > h^0(\mathscr{O}_S(H))$ such that it is not possible to apply the Alexander-Bauer lemma on these two decompositions. On the other hand, if we would apply the Alexander-Bauer lemma to the decomposition $H \equiv A + B$ then a neccessary closed condition is that $\dim |A| = 0$. But the proof of the Proposition above shows that $|A|$ is at least a pencil on $S$ since $A_i + F_i \in |A|$.

We believe that there may exist other nice decompositions for $H$ on whom we could apply our general strategy on as we have not been able to show the non-existence of this particular case.

In the next and the last chapter, we give a successful construction of the fourth divisor class of Theorem 22 not discussed in this chapter.

# 5 An explicit construction.

In this chapter we explicitly construct a smooth rational surface of degree 11 and sectional genus 8 out of one of the divisor classes in Theorem 20. This will prove our Main Theorem.

## 5.1 Proof of the main theorem.

**Theorem 39.** *Let $S$ be a rational surface with $K_S^2 = -8$. Let $\pi : S \longrightarrow \mathbb{P}^2$ denote the morphism obtained by blowing up the points $x_1, .., x_5, y_1, y_2, z_1, .., z_{10} \in \mathbb{P}^2$ and let $E_i := \pi^{-1}(x_i)$, $F_i = \pi^{-1}(y_i)$, $G_i = \pi^{-1}(z_i)$ and $L := \pi^* l$ where $l \subset \mathbb{P}^2$ is a line. It is possible to choose the points $x_1, .., x_5, y_1, y_2, z_1, .., z_{10}$ such that the divisor class*

$$H \equiv 7L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} 2F_j - \sum_{k=1}^{10} G_k$$

*is very ample on $S$ and $|H|$ embeds $S$ as a rational surface of degree 11 and sectional genus 8 in $\mathbb{P}^5$.*

*Proof.* We begin with choosing $x_1, .., x_5 \in \mathbb{P}^2$ in general position, such that:

(O1). No two points $x_i$ are infinitely near.
(O2). No three points $x_i$ are collinear.

Note that O1 and O2 are satisfied for a general choice of five points in $\mathbb{P}^2$. Let

$$\pi_1 : S_1 \longrightarrow \mathbb{P}^2$$

denote the morphism obtained by blowing up $x_1, .., x_5 \in \mathbb{P}^2$ and define $E_i := \pi^{-1}(x_i)$. On $S_1$ we study the complete linear systems associated to the following two divisor classes

$$A \equiv 6L - \sum_{i=1}^{5} 2E_i$$

$$B \equiv 4L - \sum_{i=1}^{5} E_i.$$

**Lemma 40.** *The complete linear systems $|A - K_{S_1}|$ and $|B - K_{S_1}|$ have no fixed components.*

*Proof.* Let $D_1 \equiv A - K_{S_1}$ and $D_2 \equiv B - K_{S_1}$. Suppose $|D_j|$ has a fixed components $N_j$. Let $M_j$ denote the moving parts of $|D_j|$, i.e. $|D_j - N_j| = |M_j|$. We may write $M_j \equiv a_j L - \sum_{i=1}^{5} b_{ij} E_i$. Combining Riemann-Roch and Clifford's theorem, Theorem IV.5.4 in [Har77],we have the following inequality

$$\chi(\mathscr{O}_{S_1}(D_j)) \leq \dim |D_j| = \dim |M_j|_{M_j}| + 1 \leq \frac{1}{2} M_j^2 + 1$$

or equivalently

$$\sum_i b_{ij}^2 \leq 2 + a_j^2 - 2\chi(\mathscr{O}_{S_1}(D_j))$$

by rearranging. We determine $a_j, b_{1j}, .., b_{5j} \in \mathbb{Z}_{\geq 0}$, for each choice of $1 \leq j \leq 2$.
(Case $j = 1$:) The expected dimension $\chi(\mathscr{O}_{S_1}(A - K_{S_1})) = \binom{11}{2} - 5\binom{4}{2} = 25$. Note that $a_1 \geq 7$ since $a_1 \leq 6$ would imply that $\sum b_{i1}^2 < 0$. If $a_1 = 7$, then $\sum b_{i1}^2 \leq 1$ in which case the fixed component $N_3$ would either be a double conic or a conic passing doubly through 4 points. The first possibility does not hold since $9 - a_1 = 4$ is false and the second possibility implies the first possibility. So $a_1 \neq 7$. If $a_1 = 8$, then $\sum b_{i1}^2 \leq 16$ such that $M_3$ passes through at most 1 triple point. Then $N_3$ passes through at least 4 points and contradicts O2. So $|A - K_{S_1}|$ has no fixed components.
(Case $j = 2$:) Here, $a_2^2 \leq 6$ implies that $\sum b_{i2}^2 < 0$ since $\chi(\mathscr{O}_{S_1}(B - K_{S_1})) = \binom{9}{2} - 5\binom{3}{2} = 21$. So $|B - K_{S_1}|$ has no fixed components. $\square$

**Lemma 41.** *The complete linear systems $|A|$ and $|B|$ are base-point free on $S_1$.*

*Proof.* Note that $A - K_{S_1}$ (resp. $B - K_{S_1}$) is a nef divisor. For if $A - K_{S_1}$ (resp. $B - K_{S_1}$) was not nef, then there would exist an effective divisor $D$ on $S_1$ such that $D.(A - K_{S_1}) < 0$ (resp. $D.(B - K_{S_1}) < 0$). Then $D$ would be a part of some fixed component of $|A - K_{S_1}|$ (resp. $|B - K_{S_1}|$) which would contradict Lemma 40. Now, suppose $|A|$ (resp. $|B|$) has at least one base-point. Then Reider's theorem, Theorem 1.1. in [Rei88], implies that there exists at least one effective divisor $D$ such that (1) (resp. (2)) holds:

$$(1).\ D.(A - K_{S_1}) = r \text{ and } D^2 = r - 1, \text{ for some } r \in \{0, 1\}.$$

$$(2).\ D.(B - K_{S_1}) = r \text{ and } D^2 = r - 1, \text{ for some } r \in \{0, 1\}.$$

Let $D \equiv aL - \sum_{i=1}^5 b_i E_i$. We claim there are no possibilities for $a \geq 0$ and $b_1, ..., b_5 \in \mathbb{Z}$.
Case (1): If $r = 0$, then by combining $D.(A - K_{S_1}) = 9a - \sum 3b_i = 0$ and $D^2 = a^2 - \sum b_i^2 = -1$ with the Cauchy-Schwarz inequality gives $1 + \frac{1}{9}\left(\sum b_i\right)^2 = \sum b_i^2 \leq 1 + \frac{5}{9}\sum b_i^2$. Then $\sum b_i^2 \leq \frac{9}{4} < 3$ such that $D \equiv aL - b_1E_1 - b_2E_2$, where $-1 \leq b_i \leq 1$. But the latter contradicts $\sum b_i \equiv 0(\text{mod}3)$. So $r \neq 0$. If $r = 1$, then by computing $D.(A - K_{S_1}) = 9a - \sum 3b_i = 1$ modulo 3 we get a contradiction.
Case (2): If $r = 0$, then by combining $D.(B - K_{S_1}) = 7a - \sum 2b_i = 0$ and $D^2 = a^2 - \sum b_i^2 = -1$ with the Cauchy-Schwarz inequality gives $1 + \frac{4}{49}\left(\sum b_i\right)^2 = \sum b_i^2 \leq 1 + \frac{20}{49}\sum b_i^2$. Then $\sum b_i^2 \leq \frac{49}{29} < 2$ such that we may write $D \equiv aL - b_1E_1$, where $-1 \leq b_i \leq 1$. But this contradicts $\sum 2b_i \equiv 0(\text{mod}7)$. So $r \neq 0$. If $r = 1$, then by combining $D.(B - K_{S_1}) = 7a - \sum 2b_i = 1$ and $D^2 = a^2 - \sum b_i^2 = 0$ with the Cauchy-Schwarz inequality gives $\frac{1}{2}(7a - 1) = \sum b_i \leq \sqrt{5\sum b_i^2} = a\sqrt{5}$. Then $a < 1$ such that $a = 0$ and $b_i = 0$, for all $i$. This means that $D \equiv 0$ which contradicts $D.(B - K_{S_1}) = 1$. Since there exists no effective divisor $D$ on $S_1$ as described in (1) or (2), Reider's theorem yields that $|A|$ and $|B|$ are indeed base-point free linear systems on $S_1$. $\square$

**Lemma 42.** *A general choice of curves in $|A|$ and $|B|$ are both smooth and irreducible.*

*Proof.* Let $\varphi_A : S_1 \to \mathbb{P}^{\dim|A|}$ and $\varphi_B : S_1 \to \mathbb{P}^{\dim|B|}$ denote the morphisms associated to $|A|$ and $|B|$, respectively. By Bertini's theorem, Theorem 20.2 in [BPVdV84], it suffices to only consider the cases where $\dim\varphi_A(S_1) \leq 1$ or $\dim\varphi_B(S_1) \leq 1$, i.e. when $|A|$ and $|B|$ are composed with pencils. So, suppose $\dim\varphi_A(S_1) \leq 1$ and $\dim\varphi_B(S_1) \leq 1$. Since $\dim|B| > 1$ and $S_1$ is rational surface, $B$ must neccessarily be a multiple divisor which is not the case. Therefore $\dim\varphi_B(S_1) = 2$. For $|A|$ we note that $\varphi_A$ cannot be birational since $\dim S_1 \neq \dim\varphi_A(S_1)$. Then Corollary 1.2 in [Rei88] implies that there exists a base-point free pencil $|D|$ such that $D.(A - K_{S_1}) = r$, for some $r \in \{1, 2\}$. Write $D \equiv aL - \sum_{i=1}^{5} b_i E_i$. Then $D.(A - K_{S_1}) = r$ reduces to $9a = \sum 3b_i + r$. Computing the latter modulo 3 yields $r \equiv 0(\mathrm{mod}3)$ which is clearly not the case. Therefore $S_1$ is birationally equivalent to $\varphi_A(S_1)$ and since birational equivalence preserves dimensions we get $\dim\varphi_A(S_1) = 2$. Hence, Bertini's theorem implies that a general choice of divisors in $|A|$ and $|B|$ are both smooth and irreducible. $\square$

Now we choose $y_1, y_2, z_1, .., z_{10} \in \mathbb{P}^2$ such that the complete linear systems

$$\Delta_1 = |6l - \sum 2x_i - \sum y_i - \sum z_i|$$

$$\Delta_2 = |4l - \sum x_i - \sum y_i - \sum z_i|$$

on $\mathbb{P}^2$ satisfy the following closed condition

(C1)$_1$. $\Delta_1 \neq \emptyset$ and $\Delta_2 \neq \emptyset$.
(C1)$_2$. $\Delta_{y_1,y_2} = \{(D_1, D_2) \in \Delta_1 \times \Delta_2 \mid D_1, D_2$ share common tangent directions $y_1'$ and $y_2'$ at $y_1$ and $y_2\} \neq \emptyset$.
(C1)$_3$. $\{(D_1, D_2) \in \Delta_{y_1,y_2} \mid D_1 \cap D_2 = \sum x_i + \sum y_i + \sum y_i' + \sum z_i\} \neq \emptyset$.

and that if $\pi_2 : S \longrightarrow S_1$ denotes the morphism obtained by blowing up $y_1, y_2, z_1, .., z_{10}$, where $F_i := \pi_2^{-1}(y_i)$ and $G_i = \pi_2^{-1}(z_i)$, then the following open conditions are satisfied:

(O3). $|L - \sum_{i\in I} E_i - F_r| = \emptyset$, whenever $|I| \geq 2$ and for each $1 \leq r \leq 2$.
(O4). $|2L - \sum_{i=1}^{5} E_i - F_r| = \emptyset$, for each $1 \leq r \leq 2$.
(O5). $|6L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} F_i - \sum_{k=1}^{10} G_k - 2F_r| = \emptyset$, for each $1 \leq r \leq 2$.
(O6). $|6L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} 2F_i - \sum_{k=1}^{10} G_k| = \emptyset$.
(O7). $6L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} F_i - \sum_{k=1}^{10} G_k - F_r$ is smooth, for each $1 \leq r \leq 2$.

Note that the open conditions O3-O5 are all neccessary conditions for the very ampleness of $H$, since every curve $C$ in the linear systems depicted in O3-O5 either intersects non-positive with $H$ or $p_a(C) \leq 2$ and $H.C \leq 2p_a(C)$. Now we show the following claim.

*Claim: The closed condition C1 is a non-empty condition.*

Choose a smooth and irreducible curve $A_1 \in |A|$ and consider the incidence $\Sigma \subset S_1 \times S_1 \times |B|$ given by

$$\Sigma = \{(p, q, B) \mid p, q \in A_1 \cap B,\ A_1 \text{ and } B \text{ have common tangent directions at } p \text{ and } q\}.$$

Note that $\Sigma \neq \emptyset$ since $\dim \Sigma \geq \dim |A| - 4 > 0$. By Lemma 42, we may choose $(y_1, y_2, B_1) \in \Sigma$ such that $B_1$ is smooth and irreducible on $S_1$. In particular, the curve $B_1$ is smooth at each closed point in the zero-dimensional scheme $A_1 \cap B_1$ since $|B_{|A_1}|$ is base-point free. Then we may set

$$A_1 \cap B_1 = \sum x_i + \sum y_i + \sum y_i' + \sum z_i$$

where $y_i'$ is the tangent direction of $A_1$ and $B_1$ at $y_i$, and $\{z_i\}$ are the remaining points on the intersection of $A_1$ and $B_1$. Since $|A_1|_{B_1}|$ is base-point free, there are $\#\{z_i\} = 6 \cdot 4 - 2 \cdot 5 - 4 = 10$ number of distict points in $\{z_i\}$. Now we blow up $S_1$ at the points $y_1, y_2, z_1, .., z_{10} \in S_1$. Denote

$$\pi_2 : S \longrightarrow S_1$$

as the morphism obtained by blowing up $S_1$ at $y_1, y_2, z_1, .., z_{10}$ and let $F_i := \pi_2^{-1}(y_i)$ and $G_i := \pi_2^{-1}(z_i)$. Define

$$\pi := \pi_1 \circ \pi_2 : S \longrightarrow \mathbb{P}^2$$

and denote

$$A_0 \equiv 6L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} F_j - \sum_{k=1}^{10} G_k$$

$$B_0 \equiv 4L - \sum_{i=1}^{5} E_i - \sum_{j=1}^{2} F_j - \sum_{k=1}^{10} G_k$$

as the divisor classes of the strict transforms of $A_1$ and $B_1$ on $S_1$, respectively. Then the sublinear systems $|\pi(A_0)| \subset \Delta_1$ and $|\pi(B_0)| \subset \Delta_2$ are both non-empty in $\mathbb{P}^2$, such that $\Delta_1 \neq \emptyset$ and $\Delta_2 \neq \emptyset$. Due to our construction, $(\pi(A_0), \pi(B_0)) \in \Delta_{y_1, y_2}$ and the points $x_1, .., x_5, y_1, y_2, z_1, .., z_{10}$ lie on the set-theoretic complete intersection between $\pi(A_0)$ and $\pi(B_0)$. In other words, the closed condition C1 is indeed non-empty. This proves the claim.

*Claim: The open conditions O3-O7 are non-empty.*
The conditions O3-O5 are follow from simple dimension analysis. The open condition O7 is due to Bertini's theorem.

On $S$ we study the curves $A_0$, $B_0$ and the following two divisor classes:

$$C_0 \equiv L - F_1 - F_2$$

$$H \equiv A_0 + C_0 \equiv 7L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{2} F_j - \sum_{k=1}^{10} G_k.$$

**Lemma 43.** $\mathscr{O}_{A_0}(H) \simeq \omega_{A_0}$.

*Proof.* By our construction, on $S$ we have $A_0 \cap B_0 = y_1' + y_2'$. Since tangent directions $y_1', y_2'$ at $y_1, y_2 \in S_1$ correspond to points on $F_1, F_2 \subset S$, we have $B_0|_{A_0} \equiv (F_1 + F_2)|_{A_0}$. Furthermore, since $A_0.(B_0 - F_1 - F_2) = 0$ and $B_0|_{A_0}$ is effective by construction, Lemma 28 yields that $\mathscr{O}_{A_0}(B_0 - F_1 - F_2) \simeq \mathscr{O}_{A_0}$. On the other hand, $(C_0 - K_S)|_{A_0} \equiv (B_0 - F_1 - F_2)|_{A_0}$ such that $\mathscr{O}_{A_0}(C_0 - K_S) \simeq \mathscr{O}_{A_0}$. Twisting the latter sheaves by $\mathscr{O}_{A_0}(A_0 + K_S)$ we obtain $\mathscr{O}_{A_0}(H) \simeq \mathscr{O}_{A_0}(A_0 + K_S)$. Recall that, by construction, $A_1$ was chosen as a smooth curve on $S_1$. Then $A_0$ is neccessarily smooth on $S$ such that the adjunction formula yields $\mathscr{O}_{A_0}(A_0 + K_S) \simeq \omega_{A_0}$. Hence $\mathscr{O}_{A_0}(H) \simeq \omega_{A_0}$. □

**Lemma 44.** $h^0(\mathscr{O}_S(H)) = 6$.

*Proof.* Note that $C_0^2 > 2p_a(C_0) - 2$ such that Lemma 27 tells us that $H^1(\mathscr{O}_S(C_0))$ vanishes. Taking cohomology of

$$0 \longrightarrow \mathscr{O}_S(C_0) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_{A_0}(H) \longrightarrow 0$$

we get that $h^1(\mathscr{O}_S(H)) = h^1(\mathscr{O}_{A_0}(H))$. By Lemma 43, we get $h^1(\mathscr{O}_S(H)) = h^1(\omega_{A_0}) = 1$ which combined with the expected dimension $\chi(\mathscr{O}_S(H)) = 5$ yields $h^0(\mathscr{O}_S(H)) = 6$. □

**Lemma 45.** $h^0(\mathscr{O}_S(A_0)) = 2$.

*Proof.* Taking the union of $B_0$ and the unique conic $Q$ passing through $E_1, .., E_5$, we get that $B_0 + Q \in |A_0|$. On the other hand, $A_0$ is chosen to be irreducible by construction. So we have $h^0(\mathscr{O}_S(A_0)) \geq 2$. We take cohomology of the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(K_S - A_0) \longrightarrow \mathscr{O}_S(K_S) \longrightarrow \mathscr{O}_{A_0}(K_S) \longrightarrow 0$$

and note that $h^i(\mathscr{O}_S(K_S)) = 0$ for $0 \leq i \leq 1$, since $S$ is rational. Since $h^0(\mathscr{O}_S(K_S - A_0)) = 0$, we have $h^1(\mathscr{O}_S(K_S - A_0)) = h^0(\mathscr{O}_{A_0}(K_S))$. By Lemma 43 we get $K_S|_{A_0} \equiv C_0|_{A_0}$. Combining the latter with $h^1(\mathscr{O}_S(K_S - A_0)) = h^1(\mathscr{O}_S(A_0))$ we gain $h^1(\mathscr{O}_S(A_0)) = h^0(\mathscr{O}_{A_0}(C_0))$. To compute $h^0(\mathscr{O}_{A_0}(C_0))$ we take cohomology of

$$0 \longrightarrow \mathscr{O}_S(C_0 - A_0) \longrightarrow \mathscr{O}_S(C_0) \longrightarrow \mathscr{O}_{A_0}(C_0) \longrightarrow 0$$

and note that $h^i(\mathscr{O}_S(C_0 - A_0)) = 0$ for $0 \leq i \leq 1$ since $C_0 - A_0$ is not effective on $S$ and since $(C_0 - A_0)^2 > 2p_a(C_0 - A_0) - 2$. Then $h^0(\mathscr{O}_S(C_0)) = h^0(\mathscr{O}_{A_0}(C_0))$. We have already shown that $h^1(\mathscr{O}_S(C_0)) = 0$ in the proof of Lemma 44. Therefore, $h^0(\mathscr{O}_S(C_0)) = \chi(\mathscr{O}_S(C_0)) = 1$. Combining $h^0(\mathscr{O}_S(C_0)) = h^1(\mathscr{O}_S(A_0))$ with $\chi(\mathscr{O}_S(A_0)) = 1$ we get $h^0(\mathscr{O}_S(A_0)) = 2$. □

**Lemma 46.** *The complete linear system $|H|$ restricts to a very ample linear system on $C_0$. The restriction maps $H^0(\mathscr{O}_S(H)) \longrightarrow H^0(\mathscr{O}_{C_0}(H))$ and $H^0(\mathscr{O}_S(H)) \longrightarrow H^0(\mathscr{O}_{A_0}(H))$ are surjective, for all $A_0 \in |A_0|$.*

*Proof.* The first assertion is true since $H.C_0 > 2p_a(C_0)+1$ and $\dim|C_0| = 0$. For the second assertion, consider

$$0 \longrightarrow \mathscr{O}_S(C_0) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_{A_0}(H) \longrightarrow 0$$

$$0 \longrightarrow \mathscr{O}_S(A_0) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_{C_0}(H) \longrightarrow 0.$$

Note that $h^1(\mathscr{O}_S(C_0)) = 0$ since $C_0^2 > 2p_a(C_0) - 2$. Then the first short exact sequence tells us that $H^0(\mathscr{O}_S(H)) \longrightarrow H^0(\mathscr{O}_{A_0}(H))$ is surjective, for all $A_0 \in |A_0|$. For $\alpha : H^0(\mathscr{O}_S(H)) \longrightarrow H^0(\mathscr{O}_{C_0}(H))$, by the exactness of the second short exact sequence and Lemma 45 we have $\dim\ker(\alpha) = h^0(\mathscr{O}_S(A_0)) = 2$ and by Lemma 44 we have $h^0(\mathscr{O}_S(H)) = 6$. Then $\mathrm{rank}(\alpha) = 4$. On the other hand, $h^0(\mathscr{O}_{C_0}(H)) = \chi(\mathscr{O}_{C_0}(H)) = 4$ since $H.C_0 > 2p_a(C_0) - 2$. Thus $\alpha$ is surjective. ☐

Now we proceed to show that $\mathscr{O}_{A_0}(H)$ is very ample, for all $A_0 \in |A_0|$. To show this, we partition $|A_0|$ into the following two families of curves

$$\mathscr{A}_{\text{Good}} = \{D \in |A_0| \;:\; \text{Every } A_0' \leq D \text{ satisfies } A_0'.F_i \leq 1, \text{ for all } 1 \leq i \leq 2\}$$

$$\mathscr{A}_{\text{Bad}} = \{D \in |A_0| \;:\; \exists\, A_0' \leq D \text{ satisfying } A_0'.F_i > 1, \text{ for some } 1 \leq i \leq 2\}$$

and consider each family separately.

**Lemma 47.** *$\mathscr{O}_{A_0}(H)$ is very ample, for all $A_0 \in \mathscr{A}_{Good}$.*

*Proof.* Let $A_0 \in \mathscr{A}_{\text{Good}}$ and let $A_0'$ be a subcurve of $A_0$. Denote $S' := S\backslash\{F_1, F_2, G_1, .., G_{10}\}$ and denote $S_1' := S_1\backslash\{y_1, y_2, z_1, .., z_{10}\}$. We claim that the blow-up morphism $\pi_2 : S \longrightarrow S_1$ defines an isomorphism $A_0' \simeq \pi_2(A_0')$. It is clear that $\pi_2$ yields an isomorphism $\overline{S' \cap A_0'} \simeq S_1' \cap \pi_2(A_0')$. Furthermore, we may assume $A_0'$ is irreducible, in which case $A_0' \leq A_0$ and $A_0.G_j = 1$, for all $j$, implies that $A_0'.G_j \leq 1$, for all $j$. So we may set $A_0'.F_i \leq 1$ and $A_0'.G_j \leq 1$, for all $i, j$. Note that if $A_0'.F_i < 1$ (resp. $A_0'.G_j < 1$), for some $i$ (resp. $j$), then $A_0' \cap F_i = \emptyset$ (resp. $A_0' \cap G_j = \emptyset$). Therefore, we only need to consider the exceptional divisors in $\{F_i \mid A_0'.F_i = 1\}$ and in $\{G_i \mid A_0'.G_i = 1\}$. In the latter case it is straightforward to see that $\pi_2$ and $\pi_2^{-1}$ are inverse of each other, such that we get

$$A_0 \simeq \pi_2(A_0) \equiv A_1.$$

On the other hand, since $A_1 + K_{S_1}$ does not meet any of the points $y_1, y_2, z_1, .., z_{10}$, it follows that the strict transform $A_0 + K_S$ of $A_1 + K_{S_1}$ under $\pi_2$ and the proper transform $\pi_2^*(A_1 + K_{S_1})$ of $A_1 + K_{S_1}$ under $\pi_2$ are linearly equivalent. In other words,

$$\pi_2^*\mathscr{O}_{\pi_2(A_0)}(A_1 + K_{S_1}) \simeq \mathscr{O}_{A_0}(A_0 + K_S).$$

Then $\mathscr{O}_{A_0}(A_0 + K_S)$ is very ample whenever $\mathscr{O}_{A_1}(A_1 + K_{S_1})$ is very ample. Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_{S_1}(K_{S_1}) \longrightarrow \mathscr{O}_{S_1}(A_1 + K_{S_1}) \longrightarrow \mathscr{O}_{A_0}(A_1 + K_{S_1}) \longrightarrow 0.$$

Taking cohomology and noting that $h^i(\mathscr{O}_{S_1}(K_{S_1})) = 0$, for $0 \leq i \leq 1$, we get $H^0(\mathscr{O}_{S_1}(A_1 + K_{S_1})) \simeq H^0(\mathscr{O}_{A_1}(A_1 + K_{S_1}))$ such that $\mathscr{O}_{S_1}(A_1 + K_{S_1})$ is very ample if and only if $\mathscr{O}_{A_1}(A_1 + K_{S_1})$. So it suffices to show that $\mathscr{O}_{S_1}(A_1 + K_{S_1})$ is very ample. For the latter, we decompose $A_1 + K_{S_1}$ into the following effective decomposition on $S_1$

$$D_1 \equiv L - E_5$$

$$D_2 \equiv 2L - \sum_{i=1}^{4} E_i.$$

and apply the Alexander-Bauer lemma. By Riemann-Roch, $\dim |D_j| \geq 1$ for $1 \leq j \leq 2$. Furthermore, $h^1(\mathscr{O}_{S_1}(D_j)) = 0$ since $D_j^2 > 2p_a(D_j) - 2$, for $1 \leq j \leq 2$. The latter implies that the restriction maps $H^0(\mathscr{O}_{S_1}(A_1 + K_{S_1})) \longrightarrow H^0(\mathscr{O}_{D_i}(A_1 + K_{S_1}))$ are surjective, by taking cohomology of the following short exact sequences

$$0 \longrightarrow \mathscr{O}_{S_1}(D_j) \longrightarrow \mathscr{O}_{S_1}(A_1 + K_{S_1}) \longrightarrow \mathscr{O}_{D_i}(A_1 + K_{S_1}) \longrightarrow 0,$$

for $(i, j) \in \{(1, 2), (2, 1)\}$. Now we show that the sets

$$\mathscr{S}_j = \{D'_j \mid D'_j \leq D_j \text{ for some } D_j \in |D_j|,\ (A_1 + K_{S_1}).D'_j \leq 2p_a(D'_j)\} = \emptyset,$$

for $1 \leq j \leq 2$. Let $j = 1$. Note that any subcurve $D'_1 \leq D_1$ is of the form $D'_1 \equiv aL - b_5E_5$, where $0 \leq a \leq 1$. If $a = 1$, then $(A_1 + K_{S_1}).D'_1 \leq 2p_a(D'_1)$ if and only if $b_5 \geq 3$. But the former cannot occur as that would imply that $D'_1$ would be a multiple line, which it is not. If $a = 0$, then $(A_1 + K_{S_1}).D'_1 \leq 2p_a(D'_1)$ if and only if $b_5 \geq 0$. If $b_5 = 0$, then we are done. If $b_5 > 0$, then the residual curve $D_1 - D'_1$ is effective and is of the form $L - (b_5 + 1)E_5$ which cannot occur, since $D_1$ is not a multiple line. So $\mathscr{S}_1 = \emptyset$. Let $j = 2$. Then $D'_2 \leq D_2$ is of the form $D'_2 \equiv aL - \sum_{i=1}^4 b_iE_i$, where $0 \leq a \leq 2$. If $a = 0$, then we would have $\sum b_i \geq 3$ in which case the residual curve $D_2 - D'_2$ would be a conic passing multiple through some exceptional curves, which cannot happen. If $a = 1$, then $\sum b_i \geq 3$ would contradict O2. If $a = 2$, then $(A_1 + K_{S_1}).D'_2 \leq 2p_a(D'_2)$ if and only if $6 \leq \sum b_i$, in which case $D'_2$ passes doubly through at least one point which is absurd. Thus $\mathscr{S}_j = \emptyset$, for all $1 \leq j \leq 2$, such that the Curve embedding theorem implies that $\mathscr{O}_{D_j}(H)$ is very ample, for all $1 \leq j \leq 2$. Then the Alexander-Bauer lemma implies that $\mathscr{O}_{A_1}(A_1 + K_{S_1})$ is indeed very ample. By the discussion above, this implies that $\mathscr{O}_{A_0}(A_0 + K_S) \simeq \mathscr{O}_{A_0}(H)$ is very ample. □

**Lemma 48.** *$\mathscr{O}_{A_0}(H)$ is very ample, for all $A_0 \in \mathscr{A}_{Bad}$.*

*Proof.* Let $A_0 \in \mathscr{A}_{\mathrm{Bad}}$ and let $A'_0$ be a subcurve of $A_0$ such that $A'_0.F_r \geq 2$, for some $1 \leq r \leq 2$. Then we may decompose $A_0$ into $A_0 \equiv (A_0 - nF_1 - mF_2) + nF_1 + mF_2$, for some $n, m \in \mathbb{Z}_{\geq 0}$ such that $n + m \geq 1$. In fact, the open condition O5 implies that $n, m \neq 2$. Moreover, if $n + m \geq 3$ then the inclusion map $H^0(\mathscr{O}_S(A_0 - nF_1 - mF_2)) \hookrightarrow H^0(\mathscr{O}_S(A_0 - 2F_r))$, for some $1 \leq r \leq 2$, implies that $1 \leq n + m \leq 2$ where $n, m < 2$. This leaves us with three cases, namely $(n, m) \in \{(1, 1), (1, 0), (0, 1)\}$. On the other hand, the open condition O6 yields that $(n, m) \neq (1, 1)$. So it suffices to show that $|H|$

embeds $(A_0 - F_r) + F_r$. Note that $\mathscr{O}_{F_r}(H)$ is very ample since $H.F_r \geq 2p_a(F_r) + 1$. Furthermore, the restriction maps $H^0(\mathscr{O}_S(H)) \to H^0(\mathscr{O}_{A_0-F_r}(H))$ is surjective since $F_r^2 > 2p_a(F_r) - 2$. The restriction map $H^0(\mathscr{O}_S(H)) \to H^0(\mathscr{O}_{F_r}(H))$ is also surjective since $h^0(\mathscr{O}_S(A_0 - F_r)) + h^0(\mathscr{O}_{F_r}(H)) = 6$. For the very ampleness of $\mathscr{O}_{A_0-F_r}(H)$, by O7 the curves $(A_0 - F_r)$ are smooth curves. We claim that $\mathscr{O}_{A_0-F_r}(H) \simeq \omega_{A_0-F_r}$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_S(C_0 + F_r) \longrightarrow \mathscr{O}_S(H) \longrightarrow \mathscr{O}_{A_0-F_r}(H) \longrightarrow 0.$$

Since $(C_0 + F_r)^2 > 2p_a(C_0 + F_r) - 2$ we have $h^1(\mathscr{O}_S(C_0 + F_r)) = 0$, by Lemma 27. Then $h^1(\mathscr{O}_S(H)) = h^1(\mathscr{O}_{A_0-F_r}(H)) = 1$ due to the sequence above and Lemma 44. Combining the latter with $H.(A_0 - F_r) = 2p_a(A_0 - F_r) - 2$ we conclude that $\mathscr{O}_{A_0-F_r}(H) \simeq \omega_{A_0-F_r}$ by using Lemma 28. Now, since $(A_0 - F_r)$ is smooth the adjunction formula yields that $\mathscr{O}_{A_0-F_r}(H) \simeq \mathscr{O}_{A_0-F_r}(A_0 - F_r + K_S)$ where

$$A_0 - F_r + K_S \equiv 3L - \sum_{i=1}^{5} E_i - F_r.$$

Let $D \leq A_0 - F_r + K_S$ be given by $D \equiv aL - \sum b_i E_i - cF_r$ and suppose $H.D \leq 2p_a(D)$. If $a \leq 2$, then $H.D \leq 0$ if and only if at least three of $E_1, .., E_5, F_r$ are collinear or if there exists a conic passing through $E_1, .., E_5, F_r$. Then the previous cases contradict O3 or O4, respectively. So $a = 3$. Then there are two cases, namely $0 \leq p_a(D) \leq 1$. The case $p_a(D) = 1$ is redundant, for then we would have $\sum b_i + c \geq 9$ which would imply that $p_a(D) < 1$. If $p_a(D) = 0$, then $D$ meets exactly one of the execeptional curves $E_1, .., E_5, F_r$ twice such that $H.D = 4$. So $a \neq 3$ and then the Curve embedding theorem yields that $\mathscr{O}_{A_0-F_r}(H)$ is very ample. It remains to show that $|H|$ embeds $(A_0 - F_r) \cap F_r$. Note that $\varphi_H(A_0 - F_r)$ spans a $\mathbb{P}^3$ and $\varphi_H(F_r)$ spans a $\mathbb{P}^2$. Since $\langle \varphi_H(A_0 - F_r) \cup \varphi_H(F_r) \rangle = \mathbb{P}^4$ we must have $\langle \varphi_H(A_0 - F_r) \cap \varphi_H(F_r) \rangle = L'$, where $L' = \mathbb{P}^1$. The line $L'$ meets the conic $\varphi_H(F_r)$ in exactly two points $p, q$. Since $(A_0 - F_r).F_r = 2$, the points $p, q$ are exactly the images of the points in $(A_0 - F_r) \cap F_r$. Thus $\mathscr{O}_{A_0}(H)$ is very ample. ☐

This concludes the proof of Theorem 39. ☐

A corollary of the Theorem above is the following.

**Theorem 49 (Main theorem).** *There exists linearly normal smooth rational surfaces of degree* 11 *and sectional genus* 8 *in* $\mathbb{P}^5$. *In particular, there exists a smooth rational surface with Hilbert polynomial*

$$P(n) = 11\binom{n}{2} + 4n + 1.$$

## Appendix A.

| $\deg S$ | $S$ | Linear system. |
|---|---|---|
| 1 | $\mathbb{P}^2$ | $H \equiv L$ |
| 2 | $\mathbb{F}_0$ | $H \equiv B + F$ |
| 3 | $\tilde{\mathbb{P}}^2(x_1, .., x_6)$ | $H \equiv 3L - \sum_{i=1}^{6} E_i$ |

TABLE 1: Smooth Linearly Normal Rational Surfaces in $\mathbb{P}^3$.

| $\deg S$ | $S$ | Linear system. |
|---|---|---|
| 3 | $\tilde{\mathbb{P}}^2(x_1)$ | $H \equiv 2L - E_1$ |
| 4 | $\tilde{\mathbb{P}}^2(x_1, .., x_5)$ | $H \equiv 3L - \sum_{i=1}^{5} E_i$ |
| 5 | $\tilde{\mathbb{P}}^2(x_1, .., x_8)$ | $H \equiv 4L - 2E_1 - \sum_{i=2}^{8} E_i$ |
| 6 | $\tilde{\mathbb{P}}^2(x_1, .., x_{10})$ | $H \equiv 4L - \sum_{i=1}^{10} E_i$ |
| 7 | $\tilde{\mathbb{P}}^2(x_1, .., x_{11})$ | $H \equiv 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{11} E_j$ |
| 8 | $\tilde{\mathbb{P}}^2(x_1, .., x_{11})$ | $H \equiv 7L - \sum_{i=1}^{10} 2E_i - E_{11}$ |
| 9 | $\tilde{\mathbb{P}}^2(x_1, .., x_{10})$ | $H \equiv 13L - \sum_{i=1}^{10} 4E_i$ |

TABLE 2: Smooth Non-special Linearly Normal Rational Surfaces in $\mathbb{P}^4$.

Table 2 is Theorem 1 in [Ale88].

# Appendix B.

The following are relations computed by Proposition 7 and used in Section 3.3. We have included them here to ease the readibility of some proofs in Section 3.3.

| $i$ | $H_i^2$ | $H_i.K_i$ |
|---|---|---|
| 0 | $11$ | $3$ |
| 1 | $26 - r_0$ | $12 - r_0$ |
| 2 | $59 - 3r_0 - r_1$ | $21 - r_0 - r_1$ |
| 3 | $110 - 5r_0 - 3r_1 - r_2$ | $30 - r_0 - r_1 - r_2$ |
| 4 | $179 - 7r_0 - 5r_1 - 3r_2 - r_3$ | $39 - r_0 - r_1 - r_2 - r_3$ |
| 5 | $266 - 9r_0 - 7r_1 - 5r_2 - 3r_3 - r_4$ | $48 - r_0 - r_1 - r_2 - r_3 - r_4$ |
| 6 | $371 - 11r_0 - 9r_1 - 7r_2 - 5r_3 - 3r_4 - r_5$ | $57 - r_0 - r_1 - r_2 - r_3 - r_4 - r_5$ |
| 7 | $494 - 13r_0 - 11r_1 - 9r_2 - 7r_3 - 5r_4 - 3r_5 - r_6$ | |

| $i$ | $\pi_i$ | $r_i$ |
|---|---|---|
| 0 | $8$ | $10 \leq r_0 \leq 20$ |
| 1 | $20 - r_0$ | $\lceil 9 - \frac{(12-r_0)^2}{26-r_0} \rceil \leq r_1 \leq 41 - 2r_0$ |
| 2 | $41 - 2r_0 - r_1$ | $\lceil 9 - \frac{(21-r_0-r_1)^2}{59-3r_0-r_1} \rceil \leq r_2 \leq 71 - 3r_0 - 2r_1$ |
| 3 | $71 - 3r_0 - 2r_1 - r_2$ | |
| 4 | $110 - 4r_0 - 3r_1 - 2r_2 - r_3$ | |
| 5 | $158 - 5r_0 - 4r_1 - 3r_2 - 2r_3 - r_4$ | |
| 6 | $215 - 6r_0 - 5r_1 - 4r_2 - 3r_3 - 2r_4 - r_5$ | |

## References

[Ale88] J. Alexander, *Surfaces Rationnelles Non-speciales dans* $\mathbb{P}^4$. Mathematische Zeitschrift Vol.200 n.1, p.87-110, 1988.

[AS89] E. Arrondo & I. Sols, *Classification of smooth congruences of low degree*. J. Reine Angew. Math. n.393, p.199-219, 1989.

[BPVdV84] W. Barth, C. Peters & A. Van de Ven, *Compact Complex Surfaces*. Springer Verlag, 1984.

[Bau98] I. Bauer, *The classification of surfaces in* $\mathbb{P}^5$ *having few trisecants*. Rend. Sem. Mat. Univ. Politec. Torino n.56 p.1-20, 1998.

[Bea96] A. Beauville, *Complex Algebraic Surfaces*. London Mathematical Society Student Texts, 1996.

[BS95] M.C. Beltrametti & A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*. de Gruyter Expositions in Mathematics 16, 1995.

[CF93] F. Catanese & M. Franciosi, *Divisors of small genus on algebraic surfaces and projective embeddings*. Proceedings of the conference Hirzebruch 65, Tel Aviv 1993, Contemp. Math., A.M.S. (1994), subseries 'Israel Mathematical Conference Proceedings' Vol. 9 p.109-140, 1996.

[CFHR99] F. Catanese, M. Franciosi, K. Hulek & M. Reid, *Embeddings of Curves and Surfaces*. Nagoya Mathematical Journal Vol.154 p.185-220, June 1999.

[CH97] F. Catanese & K. Hulek, *Rational surfaces in* $\mathbb{P}^4$ *with a plane curve*. Annali di Matematica Pura ed Applicata, n.172 p.229-256, 1997.

[EH87] D. Eisenbud & J. Harris, *On Varieties of Minimal Degree (A Centennial Account)*. Algebraic Geometry - Bowdoin 1985 Vol.1 p.3-13, 1987.

[Gro93] M. Gross, *Surfaces of bidegree* $(3, n)$ *in* $\mathbb{G}(1, \mathbb{P}^3)$. Math.Z.212 n.1, p.73-106, 1993.

[Har77] R. Hartshorne, *Algebraic Geometry*. Springer Verlag GTM 52, 1st edition, 1995.

[Ion82] P. Ionescu, *Embedded projective varieties of small invariants*. Algebraic Geometry, Bucharest 1982, 142-186, Lecture Notes in Math., 1056, Springer, Berlin, 1984.

[Ran88] K. Ranestad, *On Smooth Surfaces of Degree ten in the Projective Fourspace*. Dissertation Dr.philos, University of Oslo, 1988.

[Rei88] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*. Annals of Mathematics n.127 (2) p.309-316, 1988.

[Som79] A.J. Sommese, *Hyperplane sections of projective surfaces I - The adjunction mapping.* Duke Mathematical Journal n.46, p.377-401, 1979.

[Som80] A.J. Sommese, *On hyperplane sections.* Chicago Circle Conference lecture notes n.862, p.271-323, 1980.

[SV87] A.J. Sommese & A. Van de Ven, *On the adjunction mapping.* Mathematische Annalen n.278 p.593-603, 1987.

[VdV79] A. Van de Ven, *On the 2-connectedness of very ample divisors on a surface.* Duke Mathematical Journal n.46, p.403-407, 1979.